

Dual algebraic structures for the two-level pairing model

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Abstract. Duality relations are explicitly established relating the Hamiltonians and basis classification schemes associated with the number-conserving unitary and number-nonconserving quasispin algebras for the two-level system with pairing interactions. These relations are obtained in a unified formulation for both bosonic and fermionic systems, with arbitrary and, in general, unequal degeneracies for the two levels. Illustrative calculations are carried out comparing the bosonic and fermionic quantum phase transitions.

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1. Introduction

The two-level pairing model describes a finite system which undergoes a second-order quantum phase transition between weak-coupling and strong-coupling dynamical symmetry limits. This quantum phase transition is characterized by singularities in the evolution of various ground state properties as the pairing interaction strength is varied: (1) a discontinuity in the second derivative of the ground state eigenvalue, (2) a discontinuity in the first derivative of the quantum order parameter, which is defined by the relative population $\langle N_2 \rangle - \langle N_1 \rangle$ of the two levels and is analogous to the magnetization parameter in the Ising model, and (3) a vanishing energy gap Δ between the ground state and first excited state with the same conserved quantum numbers, and thus a singular level density $\rho \sim \Delta^{-1}$. Although true singularities in these quantities only occur in the limit of infinite particle number, “precursors” are found at finite N , which approach the singular limit according to definite power-law scalings [1–8]. The quantum phase transition in the two-level pairing model has long been of interest for applications to nuclei [9–11]. It has recently served as a testbed for considering phase transitional phenomena, including the finite-size scaling just described, excited state quantum phase transitions [8, 12–15], thermodynamic properties [16], decoherence [17],

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and quasidynamical symmetry [18], as well as for developing the theoretical methods for treatment of these phenomena, including continuous unitary transformation [5, 7] and Holstein-Primakoff boson expansion [19].

Finite systems with pairing-type interactions, and consisting either of bosons or fermions, occur in a broad variety of physical contexts. Fermionic examples include superconducting grains (electrons) [20] and the atomic nucleus (nucleons) [21]. Bosonic examples include the *s*-wave and *d*-wave nucleon pairs of the interacting boson model (IBM) [22], which themselves undergo a bosonic pairing interaction in the description of nuclear quadrupole collectivity, and condensates of trapped bosonic atoms [23, 24].

The Lie algebraic properties of the two-level bosonic and fermionic systems are closely parallel, but the differences which do arise fundamentally affect the irreducible representations (irreps) under which the eigenstates transform and therefore are essential to defining the spectroscopy of the system. Two complementary algebraic formulations are relevant to the description of finite pairing systems [25–30]: a *unitary* algebra is spanned by the bilinear products of a creation and annihilation operator [31, 32], and a *quasispin* algebra [25, 33, 34] is defined in terms of creation and annihilation operators for time-reversed pairs of particles. These structures are intertwined by duality relations, in particular, relating irreps of the quasispin algebra with those of an orthogonal (in the case of bosons) or symplectic (in the case of fermions) subalgebra of the full unitary algebra. Such relations have often been used [30, 35–40] to effect simplifications of the calculations for two-level and multi-level systems.

In this article, the duality relations between the unitary and quasispin algebraic structures for the two-level system are systematically established. In particular, attempts to compare results across two-level systems with different level degeneracies or between the bosonic and fermionic cases (see Ref. [14]) raise the question as to which differences in spectroscopic results are *superficial*, *i.e.*, originating from an imperfect choice of correspondence between the Hamiltonian parameters for the two cases, and which are due to more *fundamental* or irreconcilable distinctions. Therefore, a main intent of the present work is to resolve the relationships between the disparate forms of the Hamiltonian which arise in the definitions of the dynamical symmetries and in numerical studies of the transition between them. These Hamiltonians include: (1) the Casimir form defined in terms of the unitary algebra, (2) the pairing form used in studies of the fermionic system, which is essentially defined in terms of quasispin operators, and (3) the “multipole” form traditionally considered for physical reasons in bosonic studies. The relations are established in a fully general fashion which uniformly accomodates arbitrary level degeneracies (n_1 and n_2), for both the bosonic and fermionic cases.

Previous work on two-level systems has concentrated either on the so-called *s-b* boson models, with level degeneracies $n_1 = 1$ and $n_2 \geq 1$ (*i.e.*, for which one of the levels is a singlet), or on fermionic models of equal degeneracies ($n_1 = n_2$). The observations outlined here are intended to provide a foundation for more detailed future work, allowing for the most general choice of level degeneracies. The results are provided as a basis for algebraic studies of the quantum phase transitions, excited

state spectroscopic structure, and classical geometry [41, 42] of two-level and multi-level pairing models. Although the discussion is presented for two-level systems, for the sake of clarity, many of the results carry over to multi-level systems essentially without modification.

After a brief summary of the dual algebraic structures for the many-body problem in general (Sec. 2), the unitary algebraic structure is presented in detail, including categorization of the subalgebra structure, classification of the irreducible representations, construction of the generators, and identification of the Casimir operators, all in a unified form for bosonic and fermionic cases (Sec. 3). The simpler quasispin structure is also reviewed (Sec. 4). Duality relations are then established between the unitary (or Casimir) and quasispin (or pairing) formulations of the Hamiltonian (Sec. 5). These are explicitly related to the spectral properties of the two-level system through numerical calculations across the quantum phase transition, illustrating basic distinctions between the bosonic and fermionic cases, when calculated for bosonic and fermionic systems with similar level degeneracies and/or similar particle number (Sec. 6).

2. Bosonic and fermionic algebras

The fundamental Lie algebra describing transformations of a many-boson or many-fermion system is spanned by the bilinear products of creation and/or annihilation operators $a_m^\dagger a_m^\dagger$, $a_m^\dagger a_m$, and $a_{m'} a_m$ (*e.g.*, Refs. [29, 30]). For bosons, the resulting algebra is $\text{Sp}(2n, \mathbb{R})$, and for fermions it is $\text{SO}(2n)$, where m and $m' = 1, \dots, n$ range over the single particle states of the system. Two important sets of subalgebras arise: number-*conserving* subalgebras and number-*nonconserving* (quasispin) subalgebras.

The restriction to number-conserving operators, spanned by the elementary one-body operators $a_m^\dagger a_m$, constitutes a $\text{U}(n)$ algebra. The $\text{U}(n)$ algebra contains a subalgebra $\text{SO}(n)$ for the bosonic case or $\text{Sp}(n)$ for the fermionic case. If each single-particle creation operator a_m^\dagger is associated with a time-reversed partner $a_{\bar{m}}^\dagger$, these $\text{SO}(n)$ or $\text{Sp}(n)$ subalgebras are defined by the property that they leave invariant the “scalar” pair state $\sum_m a_m^\dagger a_{\bar{m}}^\dagger |0\rangle$ [30]. This special property underlies the duality relations with the quasispin pair algebra considered in the present work. More specifically, we consider rotationally-invariant problems, for which the single-particle states may be identified as the $2j + 1$ substates of single-particle levels of various angular momenta j (*i.e.*, j -shells, in the nomenclature of nuclear physics, which we adopt for either bosonic or fermionic levels). Then the creation operators are of the form $a_m^\dagger \rightarrow a_{km}^\dagger$ and $a_{\bar{m}}^\dagger \rightarrow (-)^{j_k - m} a_{k, -m}^\dagger$ for the k th level. For such rotationally-invariant systems, the $\text{SO}(n)$ or $\text{Sp}(n)$ subalgebras in turn contain the physical $\text{SO}(3) \sim \text{SU}(2)$ angular momentum algebra. Although we follow the convention of denoting the angular momentum algebra $\text{SO}(3)$ in the bosonic case and $\text{SU}(2)$ in the fermionic case, there is no material distinction between the algebras. In general, there may also be other, intervening subalgebras in the chain.

Alternatively, the scalar pair creation operator $S_+ = \frac{1}{2} \sum a_m^\dagger a_{\bar{m}}^\dagger$, the scalar pair annihilation operator $S_- = \frac{1}{2} \sum a_{\bar{m}} a_m$, and the number-conserving operator $S_0 = \frac{1}{4} \sum (a_m^\dagger a_m + \theta a_m a_m^\dagger)$ close under commutation, where $\theta = +$ for bosonic systems or $\theta = -$ for fermionic systems. These operators define a number-nonconserving pair quasispin algebra, either $\mathcal{SU}(1,1)$ for bosons or $\mathcal{SU}(2)$ for fermions. The caligraphic notation for the quasispin algebras is adopted [30] to avoid ambiguity between the $\mathcal{SU}(2)$ quasispin algebra and the $SU(2)$ angular momentum algebra.

In summary, for a bosonic system, the subalgebras under consideration are

$$\mathrm{Sp}(2n, \mathbb{R}) \supset \begin{cases} \mathrm{U}(n) \supset \mathrm{SO}(n) \supset \cdots \supset \mathrm{SO}(3) \\ \mathcal{SU}(1,1) \supset \mathrm{U}(1), \end{cases} \quad (1)$$

and, for a fermionic system, they are

$$\mathrm{SO}(2n) \supset \begin{cases} \mathrm{U}(n) \supset \mathrm{Sp}(n) \supset \cdots \supset \mathrm{SU}(2) \\ \mathcal{SU}(2) \supset \mathrm{U}(1). \end{cases} \quad (2)$$

The subalgebras of $\mathrm{U}(n)$ are useful in the classification of states not only for the pairing Hamiltonian (defined in Sec. 5.2) but also for a much richer range of Hamiltonians [43].

A close relation between unitary chain and quasispin subalgebras arises since the quasispin and orthogonal or symplectic algebras may be embedded as mutually commuting “dual” algebras within the larger $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}(2n)$ algebra. The algebraic foundations are discussed in detail in Refs. [25–30]. Here we simply note that duality denotes the situation in which the states within a space may be classified simultaneously in terms of two mutually commuting groups (or algebras) G_1 and G_2 , such that the irrep labels of the two groups are in one-to-one correspondence. For the present problem, the embedding and associated labels are given, for the bosonic case, by

$$\mathrm{Sp}(2n, \mathbb{R}) \supset \underset{[v]}{[\mathrm{SO}(n) \supset \cdots \supset \mathrm{SO}(3)]} \underset{J}{\otimes} \underset{S}{[\mathcal{SU}(1,1) \supset \mathrm{U}(1)]} \underset{N}{} \quad (3)$$

and, for the fermionic case, by

$$\mathrm{SO}(2n) \supset \underset{\{v\}}{[\mathrm{Sp}(n) \supset \cdots \supset \mathrm{SU}(2)]} \underset{J}{\otimes} \underset{S}{[\mathcal{SU}(2) \supset \mathrm{U}(1)]} \underset{N}{} \quad (4)$$

The seniority label v (Sec. 3.2) and quasispin label S (Sec. 4) are in one-to-one correspondence, *i.e.*, specifying the value of one uniquely determines the value of the other, and *vice versa*.

The duality relations hold equally well regardless of whether the n -dimensional single-particle space is construed to consist of a single j -shell ($n = 2j + 1$, odd for bosons or even for fermions), two j -shells ($n = n_1 + n_2$), or, indeed, multiple j -shells. The one-level case has been considered in detail (*e.g.*, Ref. [29]). However, we find that the detailed construction of operators for two-level systems within the context of this duality, as needed for spectroscopic studies of these systems, requires elaboration. Although, for simplicity, we consider only the case of two levels, the results may readily be generalized to additional levels.

3. Unitary algebra

3.1. Subalgebra chains

Consider the $U(n)$ subalgebra chains for the two-level system, consisting of either bosonic or fermionic levels, of possibly unequal degeneracies. If the levels are j -shells of angular momenta j_1 and j_2 , the level degeneracies are $n_1 = 2j_1 + 1$ and $n_2 = 2j_2 + 1$, and the total degeneracy of the system is $n = n_1 + n_2$. For the bosonic case, we have

$$U(n_1 + n_2) \supset \left\{ \begin{array}{c} SO(n_1 + n_2) \\ [v] \\ U_1(n_1) \otimes U_2(n_2) \\ [N_1] \quad [N_2] \end{array} \right\} \supset SO_1(n_1) \otimes SO_2(n_2) \supset SO_1(3) \otimes SO_2(3) \supset SO_{12}(3) \quad (5)$$

$[v_1] \quad [v_2] \quad J_1 \quad J_2 \quad J$

and, for the fermionic case, we have

$$U(n_1 + n_2) \supset \left\{ \begin{array}{c} Sp(n_1 + n_2) \\ \{v\} \\ U_1(n_1) \otimes U_2(n_2) \\ \{N_1\} \quad \{N_2\} \end{array} \right\} \supset Sp_1(n_1) \otimes Sp_2(n_2) \supset SU_1(2) \otimes SU_2(2) \supset SU_{12}(2), \quad (6)$$

$\{v_1\} \quad \{v_2\} \quad J_1 \quad J_2 \quad J$

where $n_1 = 2j_1 + 1$ and $n_2 = 2j_2 + 1$. The irreducible representation labels, indicated beneath the symbol for each algebra, are defined in Sec. 3.2, and the algebras themselves are constructed explicitly in Sec. 3.3. Throughout the following discussion, bosonic and fermionic cases will be considered in parallel.

The subalgebras summarized in (5) and (6) are generically present, regardless of the level degeneracies n_1 and n_2 , for n_1 and $n_2 \geq 2$. However, several clarifying comments are in order:

(1) The important special case of a singlet bosonic level ($j_k = 0$) leads to $n_k = 1$, and the corresponding orthogonal algebra $SO_k(n_k)$ is undefined. The label v_k may still be defined, in a limited sense, through the quasispin, as noted in Sec. 4. Two-level bosonic problems in which $j_1 = 0$ are termed *s-b* boson models. These include the Schwinger boson realization ($j_1 = j_2 = 0$) of the Lipkin model [44]. The subalgebra chains and labeling schemes for the *s-b* models were considered in Ref. [14].

(2) For a fermionic level with $j_k = \frac{1}{2}$, and therefore $n_k = 2$, the symplectic algebra $Sp_k(n_k)$ in (6) is identical to the $SU_k(2)$ angular momentum algebra.

(3) Additional subalgebras of $U(n_1 + n_2)$ may also arise, parallel to chains indicated above and still containing the angular momentum algebra, *e.g.*, for the interacting boson model ($n_1 = 1$ and $n_2 = 5$), there is a physically relevant chain $U(6) \supset SU(3) \supset SO_{12}(3)$ [22]. However, since these chains are not directly relevant to the pairing problem and cannot be treated in a uniform fashion for arbitrary n_1 and n_2 , they are not considered further here.

(4) Further subalgebras may also intervene between $SO(n)$ and $SO(3)$, or between $Sp(n)$ and $SU(2)$, the classic example being the appearance of the exceptional algebra

G_2 in the chain $SO(7) \supset G_2 \supset SO(3)$ [31].

(5) Whenever two realizations of the same algebra commute with each other, the sum generators $G_i = G_i^{(1)} + G_i^{(2)}$ also form a new realization of the algebra, as in ordinary angular momentum addition. For instance, addition of the angular momentum generators for the two levels $[L = L^{(1)} + L^{(2)}]$ gives the generators of the sum angular momentum algebra $SO_{12}(3)$ (bosonic) or $SU_{12}(2)$ (fermionic) in (5) and (6). However, more generally, if the two levels have equal degeneracies ($n_1 = n_2 \equiv n_{12}$), such a combination of generators may also be made higher in the subalgebra chains (5) and (6), yielding $U_1(n_{12}) \otimes U_2(n_{12}) \supset U_{12}(n_{12}) \supset SO_{12}(3)$ and $SO_1(n_{12}) \otimes SO_2(n_{12}) \supset SO_{12}(n_{12}) \supset SO_{12}(3)$ for the bosonic case, or similarly $U_1(n_{12}) \otimes U_2(n_{12}) \supset U_{12}(n_{12}) \supset SU_{12}(2)$ and $Sp_1(n_{12}) \otimes Sp_2(n_{12}) \supset Sp_{12}(n_{12}) \supset SU_{12}(2)$ for the fermionic case.

3.2. Branching

The branching rules for the irreps arising in the bosonic or fermionic realizations of the algebras in (5) and (6) provide the classification of states for the two-level pairing model. Some, but not all, of these branchings can be expressed in closed form.

For the *bosonic* realization of $U(n)$, the symmetric irreps $[N] \equiv [N0 \dots 0]$ (with n labels) are obtained, where N is the occupation number. For $SO(n)$, the irreps are $[v] \equiv [v0 \dots 0]$ (with $\lfloor n/2 \rfloor$ labels).

The $U(n) \rightarrow SO(n)$ branching is of the type considered by Hammermesh [45] and is given by

$$v = (N \bmod 2), \dots, N - 2, N. \quad (7)$$

This rule applies both to the branching $U(n_1 + n_2) \rightarrow SO(n_1 + n_2)$ and to the branching associated with each of the two levels in $U_1(n_1) \otimes U_2(n_2) \rightarrow SO_1(n_1) \otimes SO_2(n_2)$. Note that n is odd for a single bosonic level and is even for the two-level system. [The $U(n) \rightarrow U_1(n_1) \otimes U_2(n_2)$ branching rule follows trivially from additivity of the number operators, $N = N_1 + N_2$.]

For the branching $SO(n_1 + n_2) \rightarrow SO_1(n_1) \otimes SO_2(n_2)$, the allowed v_1 and v_2 are obtained by considering all partitions of v as

$$v = v_1 + v_2 + 2n_v \quad (n_v = 0, 1, \dots, \lfloor v/2 \rfloor). \quad (8)$$

This rule may be verified by dimension counting arguments, that is, $\dim[v] = \sum_{v_1 v_2} \dim[v_1] \dim[v_2]$, using the Weyl dimension formula [45]. Notice that for the bosonic system (in contrast to the fermionic case below) the branching rule for $SO(n_1 + n_2) \rightarrow SO_1(n_1) \otimes SO_2(n_2)$ is independent of the level degeneracies n_1 and n_2 , and the total occupation number N influences the allowed $SO_1(n_1)$ and $SO_2(n_2)$ irreps only through the constraint (7) on v .

The allowed partitions $v \rightarrow (v_1 v_2)$ are listed, for low v , in Table 1. As a concrete example, for $N = 2$, the allowed $SO(n_1 + n_2)$ irreps have $v = 0$ and 2, with branchings to $SO_1(n_1) \otimes SO_2(n_2)$ given by the corresponding rows of Table 1. As a specific example of the equivalence of dimensions, consider the case of the two-level bosonic system with

Table 1. Partitions $v \rightarrow (v_1, v_2)$ of the form permitted by the $\text{SO}(n_1 + n_2) \rightarrow \text{SO}_1(n_1) \otimes \text{SO}_2(n_2)$ branching rule (8). The partitions permitted by the $\text{Sp}(n_1 + n_2) \rightarrow \text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$ branching rule (10) are a subset of these.

v	n_v	(v_1, v_2)
0	0	(0,0)
1	0	(1,0), (0,1)
2	0	(2,0), (1,1), (0,2)
	1	(0,0)
3	0	(3,0), (2,1), (1,2), (0,3)
	1	(1,0), (0,1)
4	0	(4,0), (3,1), (2,2), (1,3), (0,4)
	1	(2,0), (1,1), (0,2)
	2	(0,0)
5	0	(5,0), (4,1), (3,2), (2,3), (1,4), (0,5)
	1	(3,0), (2,1), (1,2), (0,3)
	2	(1,0), (0,1)
6	0	(6,0), (5,1), (4,2), (3,3), (2,4), (1,5), (0,6)
	1	(4,0), (3,1), (2,2), (1,3), (0,4)
	2	(2,0), (1,1), (0,2)
	3	(0,0)

$j_1 = j_2 = 1$, thus described by $\text{SO}(6) \supset \text{SO}_1(3) \otimes \text{SO}_2(3)$. The $v = 2$ irrep of $\text{SO}(6)$ has dimension 20, while the corresponding $\text{SO}(3) \otimes \text{SO}(3)$ irreps likewise have total dimension $\dim(2, 0) + \dim(1, 1) + \dim(0, 2) + \dim(0, 0) = (5)(1) + (3)(3) + (1)(5) + (1)(1) = 20$.

The branchings of the form $\text{SO}(n) \rightarrow \text{SO}(3)$, needed for $\text{SO}_1(n_1) \otimes \text{SO}_2(n_2) \rightarrow \text{SO}_1(3) \otimes \text{SO}_2(3)$, are more complicated and, in general, involve missing labels. However, such branchings occur widely in physical applications, and general methods exist for the solution based on weights or character theory [29, 46]. An explicit multiplicity formula is obtained in Ref. [47], applicable to the symmetric irreps arising in the present bosonic case (5). Finally, the reduction $\text{SO}_1(3) \otimes \text{SO}_2(3) \rightarrow \text{SO}_{12}(3)$ is governed by the usual triangle inequality for angular momentum addition.

The branching rules for the *fermionic* case are nearly identical, with a few modifications. For $\text{U}(n)$ we obtain the *antisymmetric* irreps $\{N\} \equiv [1, \dots, 1, 0, \dots, 0]$, that is, with N unit entries (out of n labels total). Similarly, for $\text{Sp}(n)$ we have $\{v\} \equiv [1, \dots, 1, 0, \dots, 0]$, with v unit entries (out of $n/2$ labels total).

The branching rule for $\text{U}(n) \rightarrow \text{Sp}(n)$ [45] requires the modification of (7) to

$$v = (N' \bmod 2), \dots, N' - 2, N', \quad (9)$$

where $N' \equiv \min(N, n - N)$. Notice, therefore, that $v \leq \frac{1}{2}n$. This rule applies to $\text{U}(n_1 + n_2) \rightarrow \text{Sp}(n_1 + n_2)$ and to $\text{U}_1(n_1) \otimes \text{U}_2(n_2) \rightarrow \text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$.

The values of v_1 and v_2 arising in the branching $\text{Sp}(n_1 + n_2) \rightarrow \text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$ are again given by the partitioning condition (8), but now subject to an additional

Table 2. Branching $\text{Sp}(4) \rightarrow \text{Sp}_1(2) \otimes \text{Sp}_2(2)$ for irreps of the two-level fermionic system consisting of two $j = 1/2$ levels. Allowed irreps are a subset of those listed in Table 1. The Cartan labels of these irreps with respect to the isomorphic chain $\text{SO}(5) \supset \text{SO}(4)$ are also indicated.

N	v	$[\lambda_1, \lambda_2]_{\text{SO}(5)}$	(v_1, v_2)	$[\lambda_1, \lambda_2]_{\text{SO}(4)}$
0	0	[0,0]	(0,0)	[0,0]
1	1	$[\frac{1}{2}, \frac{1}{2}]$	(1,0), (0,1)	$[\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, -\frac{1}{2}]$
2	0	[0,0]	(0,0)	[0,0]
	2	[1,0]	(1,1), (0,0)	[1,0], [0,0]
3	1	$[\frac{1}{2}, \frac{1}{2}]$	(1,0), (0,1)	$[\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, -\frac{1}{2}]$
4	0	[0,0]	(0,0)	[0,0]

Table 3. Branching $\text{Sp}(20) \rightarrow \text{Sp}_1(10) \otimes \text{Sp}_2(10)$ for irreps of the two-level fermionic system consisting of two $j = 9/2$ levels, with $N = 10$. The fermionic branching rule (10) restricts the allowed irreps, relative to those listed in Table 1, for $v > \frac{1}{2}n_{12} = 5$.

v	$n_{12} - v$	(v_1, v_2)
0		(0,0)
2		(2,0), (1,1), (0,2), (0,0)
4		(4,0), (3,1), (2,2), (1,3), (0,4), (2,0), (1,1), (0,2), (0,0)
6	4	(5,1), (4,2), (3,3), (2,4), (1,5), (4,0), (3,1), (2,2), (1,3), (0,4), (2,0), (1,1), (0,2), (0,0)
8	2	(5,3), (4,4), (3,5), (4,2), (3,3), (2,4), (3,1), (2,2), (1,3), (2,0), (1,1), (0,2), (0,0)
10	0	(5,5), (4,4), (3,3), (2,2), (1,1), (0,0)

constraint, so

$$\begin{aligned} v &= v_1 + v_2 + 2n_v \quad (n_v = 0, 1, \dots, \lfloor v/2 \rfloor) \\ |(v_1 - v_2) - \frac{1}{2}(n_1 - n_2)| &\leq \frac{1}{2}(n_1 + n_2) - v, \end{aligned} \quad (10)$$

as can again be verified by dimensional counting. The branching rules (9) and (10) together automatically enforce $v_1 \leq \frac{1}{2}n_1$ and $v_2 \leq \frac{1}{2}n_2$. For two levels of *equal* degeneracy ($n_1 = n_2 \equiv n_{12}$), the constraint simplifies to

$$|v_1 - v_2| \leq n_{12} - v \quad (11)$$

and only serves to exclude (v_1, v_2) values when $v > \frac{1}{2}n_{12}$.

For illustration, branchings for the low-dimensional case $\text{Sp}(4) \rightarrow \text{Sp}(2) \otimes \text{Sp}(2)$ (two $j = \frac{1}{2}$ levels) are given in Table 2. The chain $\text{Sp}(4) \supset \text{Sp}(2) \otimes \text{Sp}(2)$ is isomorphic to the canonical chain $\text{SO}(5) \supset \text{SO}(4)$ of orthogonal algebras, and the branchings given in Table 2 therefore also follow from the $\text{SO}(n)$ canonical branching rule [48–50]. The $\text{SO}(5)$ Cartan labels are given by $[\lambda_1, \lambda_2] = [\frac{1}{2}(\lambda'_1 + \lambda'_2), \frac{1}{2}(\lambda'_1 - \lambda'_2)]$, where $[\lambda'_1, \lambda'_2]$ are the $\text{Sp}(4)$ Cartan labels, and the $\text{SO}(4)$ Cartan labels are given by $[\lambda_1, \lambda_2] = [\frac{1}{2}(v_1 + v_2), \frac{1}{2}(v_1 - v_2)]$ (see Ref. [51] for a summary of notation for this chain). Branchings for the higher-dimensional case $\text{Sp}(20) \supset \text{Sp}(10) \otimes \text{Sp}(10)$ are given in Table 3.

3.3. Generators

Of the generators for the algebras in chains (5) and (6), those involving a single level are well known [32]. Here we must construct the generators for the two-level system. Since the subalgebra chains terminate in the two-level angular momentum algebra, $\text{SO}_{12}(3) \sim \text{SU}_{12}(2)$, it is most natural to express the generators as spherical tensors with respect to this angular momentum algebra.

First, let us briefly review the results for the single j -shell, with creation operators a_m^\dagger and annihilation operators a_m ($m = -j, -j+1, \dots, j$). In the case of a single *bosonic* level, with degeneracy $n = 2j+1$ (j integer), the subalgebra chain for $\text{U}(n)$ is $\text{U}(n) \supset \text{SO}(n) \supset \dots \supset \text{SO}(3)$ [see (1)]. The generators of $\text{U}(n)$, in spherical tensor form, are the bilinears

$$G_\gamma^{(g)} = (a^\dagger \times \tilde{a})_\gamma^{(g)} \quad (g = 0, 1, \dots, 2j), \quad (12)$$

where $\gamma = -g, -g+1, \dots, +g$. The product of two spherical tensor operators is defined by $(A^a \times B^b)_\gamma^c = \sum_{\alpha\beta} (a\alpha b\beta | c\gamma) A_\alpha^a B_\beta^b$. We follow the time reversal phase convention $\tilde{A}_\alpha^a = (-)^{a-\alpha} A_{-\alpha}^a$ [52]. Thus, *e.g.*, $\tilde{a}_m = (-)^{j-m} a_{-m}$, where the time reversal phase factor is required for the annihilation operator to transform as a spherical tensor under rotation.§

The commutators of the generators are most conveniently expressed in the spherical tensor coupled form (see appendix)

$$[G^{(e)}, G^{(f)}]^{(g)} = (-)^g [1 - (-)^{e+f+g}] \hat{e} \hat{f} \left\{ \begin{matrix} e & f & g \\ j & j & j \end{matrix} \right\} G^{(g)}, \quad (13)$$

where we adopt the shorthand $\hat{j} = (2j+1)^{1/2}$. The coefficient on the right hand side of (13) vanishes unless $e+f+g$ is odd. Consequently, the generators $G_\gamma^{(g)}$ with g odd ($g = 1, 3, \dots, 2j-1$) close under commutation, forming the basis for the subalgebra $\text{SO}(n)$. Finally, the generators $G_\gamma^{(1)}$, which span the $\text{SO}(3)$ algebra, are proportional to the physical angular momentum generators $L_\lambda^{(1)} = [\frac{1}{3}j(j+1)(2j+1)]^{1/2} (a^\dagger \times \tilde{a})_\lambda^{(1)}$ for a single bosonic j -shell.

For a single *fermionic* level, with degeneracy $n = 2j+1$ (j half-integer), we have instead the chain $\text{U}(n) \supset \text{Sp}(n) \supset \dots \supset \text{SU}(2)$ [see (2)]. The generators $G_\gamma^{(g)}$ of $\text{U}(n)$ again obey the commutation relations (13), and the generators with g odd ($g = 1, 3, \dots, 2j$) now span the $\text{Sp}(n)$ algebra. The $G_\gamma^{(1)}$ are proportional to the physical angular momentum operators, now given by $L_\lambda^{(1)} = -[\frac{1}{3}j(j+1)(2j+1)]^{1/2} (a^\dagger \times \tilde{a})_\lambda^{(1)}$, closing as an $\text{SU}(2)$ algebra.

Proceeding now to the algebras involving both levels of the two-level system, let us reduce the complexity of the subscripts, relative to the generic multi-level notation a_{km}^\dagger , by denoting the creation operators for the two levels by $a_\alpha^\dagger (\equiv a_{1,\alpha}^\dagger)$ and $b_\beta^\dagger (\equiv a_{2,\beta}^\dagger)$, respectively, with angular momenta $j_a \equiv j_1$ and $j_b \equiv j_2$, where $\alpha = -j_a, -j_a+1, \dots$,

§ The convention $\tilde{A}_\alpha^{(a)} = (-)^{a+\alpha} A_{-\alpha}^{(a)}$ also arises in the literature. The relative sign between these conventions implies straightforward modifications $\tilde{A}^{(a)} \rightarrow (-)^{2a} \tilde{A}^{(a)}$ to signs throughout the following results.

Table 4. Commutation relations for the generators of the $U(n_1 + n_2)$ algebra of the two-level *bosonic* system ($\theta = +$) or *fermionic* system ($\theta = -$), in coupled form.

$E^{(e)}$	$F^{(f)}$	$[E^{(e)}, F^{(f)}]^{(g)}$
$G_{aa}^{(e)}$	$G_{aa}^{(f)}$	$(-)^g [1 - (-)^{e+f+g}] \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_a & j_a \end{smallmatrix} \right\} G_{aa}^{(g)}$
$G_{bb}^{(e)}$	$G_{bb}^{(f)}$	$(-)^g [1 - (-)^{e+f+g}] \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_b & j_b \end{smallmatrix} \right\} G_{bb}^{(g)}$
$G_{ab}^{(e)}$	$G_{ab}^{(f)}$	0
$G_{ba}^{(e)}$	$G_{ba}^{(f)}$	0
$G_{aa}^{(e)}$	$G_{bb}^{(f)}$	0
$G_{ab}^{(e)}$	$G_{bb}^{(f)}$	$-(-)^{e+f} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_a & j_b \end{smallmatrix} \right\} G_{aa}^{(g)} + (-)^g \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_b & j_a \end{smallmatrix} \right\} G_{bb}^{(g)}$
$G_{aa}^{(e)}$	$G_{ab}^{(f)}$	$\theta (-)^{j_a+j_b+g} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_a & j_a \end{smallmatrix} \right\} G_{ab}^{(g)}$
$G_{aa}^{(e)}$	$G_{ba}^{(f)}$	$-\theta (-)^{j_a+j_b+e+f} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_a & j_a \end{smallmatrix} \right\} G_{ba}^{(g)}$
$G_{bb}^{(e)}$	$G_{ab}^{(f)}$	$-\theta (-)^{j_a+j_b+e+f} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_b & j_b \end{smallmatrix} \right\} G_{ab}^{(g)}$
$G_{bb}^{(e)}$	$G_{ba}^{(f)}$	$\theta (-)^{j_a+j_b+g} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_b & j_b \end{smallmatrix} \right\} G_{ba}^{(g)}$

$+j_a$ and $\beta = -j_b, -j_b + 1, \dots, +j_b$. The level degeneracies appearing in the algebra labels are $n_1 = 2j_a + 1$ and $n_2 = 2j_b + 1$. The algebra $U(n_1 + n_2)$ is spanned by

$$\begin{aligned}
G_{aa}^{(g)} &= (a^\dagger \times \tilde{a})^{(g)} & (g = 0, 1, \dots, 2j_a) \\
G_{ab}^{(g)} &= (a^\dagger \times \tilde{b})^{(g)} & (g = |j_a - j_b|, \dots, j_a + j_b) \\
G_{ba}^{(g)} &= (b^\dagger \times \tilde{a})^{(g)} & (g = |j_a - j_b|, \dots, j_a + j_b) \\
G_{bb}^{(g)} &= (b^\dagger \times \tilde{b})^{(g)} & (g = 0, 1, \dots, 2j_b).
\end{aligned} \tag{14}$$

The commutation relations for these generators are listed in Table 4. They may all be obtained from the general bilinear commutation relation (A.9). Notice the nearly identical commutation relations for the bosonic and fermionic realizations of the $U(n_1 + n_2)$ algebra, with sign differences indicated by the presence of the symbol θ in Table 4 (recall that $\theta = +$ for the bosonic case and $\theta = -$ for the fermionic case). Commutators not listed in Table 4, *e.g.*, $[G_{ab}^{(e)}, G_{ba}^{(f)}]^{(g)}$, can be obtained from those given, by the coupled commutator symmetry relation (A.4). The subalgebra $U_1(n_1) \otimes U_2(n_2)$ is obtained by simply omitting the “mixed” generators $G_{ab}^{(g)}$ and $G_{ba}^{(g)}$.

The $SO(n_1 + n_2)$ subalgebra, for the bosonic case, or $Sp(n_1 + n_2)$ subalgebra, for the fermionic case, is then obtained by restricting (14) to the following generators: $G_{aa}^{(g)}$ with g odd ($g = 1, 3, \dots, 2j_a - 1$ or $2j_a$), $G_{bb}^{(g)}$ with g odd ($g = 1, 3, \dots, 2j_b - 1$ or $2j_b$), and certain linear combinations of the form $F^{(g)} = \eta_g G_{ab}^{(g)} + \xi_g G_{ba}^{(g)}$, the coefficients of which are determined by the requirement of closure. Specifically, using the results of Table 4, it is found that closure is obtained if $\xi_g/\eta_g = \sigma_0(-)^g$ for all g . That is, the relative sign between the terms must alternate, between generators $F^{(g)}$ with even and odd tensor rank g , but an overall sign parameter σ_0 may be chosen as either ± 1 . Thus, we obtain

$$F^{(g)} = \eta_g [(a^\dagger \times \tilde{b})^{(g)} + \sigma_0(-)^g (b^\dagger \times \tilde{a})^{(g)}]. \tag{15}$$

Note therefore that there are actually *two* distinct $SO(n_1 + n_2)$ subalgebras which may

Table 5. Commutation relations for the generators of the $\text{SO}(n_1 + n_2)$ algebra of the two-level *bosonic* system ($\theta = +$) or the $\text{Sp}(n_1 + n_2)$ algebra of the *fermionic* system ($\theta = -$), in coupled form.

$E^{(e)}$	$F^{(f)}$	$[E^{(e)}, F^{(f)}]^{(g)}$
$G_{aa}^{(e)}$	$G_{aa}^{(f)}$	$2(-)^g \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_a & j_a \end{smallmatrix} \right\} G_{aa}^{(g)}$
$G_{bb}^{(e)}$	$G_{bb}^{(f)}$	$2(-)^g \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_b & j_b \end{smallmatrix} \right\} G_{bb}^{(g)}$
$F^{(e)}$	$F^{(f)}$	$-2(-)^{j_a+j_b+f} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_a & j_b \end{smallmatrix} \right\} G_{aa}^{(g)} - 2(-)^{j_a+j_b+e} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_b & j_a \end{smallmatrix} \right\} G_{bb}^{(g)}$
$G_{aa}^{(e)}$	$G_{bb}^{(f)}$	0
$G_{aa}^{(e)}$	$F^{(f)}$	$\theta(-)^{j_a+j_b+g} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_b & j_a & j_a \end{smallmatrix} \right\} F^{(g)}$
$G_{bb}^{(e)}$	$F^{(f)}$	$\theta(-)^{j_a+j_b+f} \hat{e} \hat{f} \left\{ \begin{smallmatrix} e & f & g \\ j_a & j_b & j_b \end{smallmatrix} \right\} F^{(g)}$

be included in (5), or two $\text{Sp}(n_1 + n_2)$ subalgebras in chain (6), distinguished by the relative sign σ_0 in the generators $F^{(g)}$.

An overall arbitrary phase η_g remains in the definition of $F^{(g)}$. If we choose $F^{(g)}$ to be a “self-adjoint” tensor, in the sense that

$$F^{(g)\dagger} = \tilde{F}^{(g)}, \quad (16)$$

i.e., $F_{\gamma}^{(g)\dagger} = (-)^{g-\gamma} F_{-\gamma}^{(g)}$, the commutation relations among the generators take on a simple form and, moreover, involve only real coefficients. Let $\sigma_0 = (-)^s$, with $s = 0$ or 1. Then a self-adjoint tensor $F^{(g)}$ is obtained for the choice

$$\eta_g = \begin{cases} 1 & a + b + s \text{ even} \\ i & a + b + s \text{ odd.} \end{cases} \quad (17)$$

The commutation relations for the $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ generators are listed in Table 5.

The phase choice (17) for the two-level generator $F^{(g)}$ also offers consistency with the s - b boson models, where $F^{(j_b)}$ plays an important role as a physical transition operator. For instance, in the IBM ($j_s = 0$ and $j_d = 2$), the choice $\sigma_0 = +$ (*i.e.*, $s = 0$) gives $\text{SO}(6)$ generator $F^{(2)} = (s^\dagger \times \tilde{d})^{(2)} + (d^\dagger \times \tilde{s})^{(2)}$, which is the leading-order electric quadrupole operator [38]. The choice $\sigma_0 = -$ (*i.e.*, $s = 1$) instead yields the generator $F^{(2)} = i[(s^\dagger \times \tilde{d})^{(2)} - (d^\dagger \times \tilde{s})^{(2)}]$ of a distinct $\text{SO}(6)$ subalgebra, denoted by $\overline{\text{SO}}(6)$ [53], which has been shown to be relevant to the decomposition of nuclear excitations into intrinsic and collective parts [54].

Finally, construction of the remaining subalgebras in (5) and (6) follows by application of the same principles. The $\text{SO}_1(n_1) \otimes \text{SO}_2(n_2)$ or $\text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$ algebra is obtained by restriction to $G_{aa}^{(g)}$ and $G_{bb}^{(g)}$ with g odd, and $\text{SO}_1(3) \otimes \text{SO}_2(3)$ or $\text{SU}_1(2) \otimes \text{SU}_2(2)$ is obtained by further restriction to $g = 1$. The combined angular momentum algebra, $\text{SO}_{12}(3)$ or $\text{SU}_{12}(2)$, then has generators $L_\lambda^{(1)}$, where

$$L^{(1)} = \theta[\frac{1}{3}j_a(j_a + 1)(2j_a + 1)]^{1/2} G_{aa}^{(1)} + \theta[\frac{1}{3}j_b(j_b + 1)(2j_b + 1)]^{1/2} G_{bb}^{(1)}. \quad (18)$$

Table 6. Casimir operator eigenvalue formulas relevant to the two-level pairing model. Expressions are given first in terms of the generic Cartan irrep labels $[\lambda_1 \lambda_2 \dots \lambda_k]$ and then specialized to the specific irreps arising for the two-level pairing system.

Algebra	$\langle C_2 \rangle_{[\lambda_1 \lambda_2 \dots]}$	Irrep	$\langle C_2 \rangle$	System
SO(n) $n = 2k + 1$	$\sum_{i=1}^k 2\lambda_i(\lambda_i + 2k + 1 - 2i)$	$[v]$	$2v(v + n - 2)$	Bosonic single-level
SO(n) $n = 2k$	$\sum_{i=1}^k 2\lambda_i(\lambda_i + 2k - 2i)$	$[v]$	$2v(v + n - 2)$	Bosonic two-level
Sp(n) $n = 2k$	$\sum_{i=1}^k 2\lambda_i(\lambda_i + 2k + 2 - 2i)$	$\{v\}$	$2v(-v + n + 2)$	Fermionic
U(n)	$\sum_{i=1}^n \lambda_i(\lambda_i + n + 1 - 2i)$	$[N]$	$N(N + n - 1)$	Bosonic
		$\{N\}$	$N(-N + n + 1)$	Fermionic

3.4. Casimir operators

To exploit the symmetry properties of the two-level pairing model with respect to the subalgebras of $U(n_1 + n_2)$, it will be necessary (Sec. 5.1) to express the Hamiltonian in terms of the quadratic Casimir operators of the algebras in (5) and (6). Identification of the Casimir operator proceeds in two stages. First, a quadratic operator which commutes with the generators must be identified. This only defines the Casimir operator to within a normalization (and phase) factor. It is then desirable to choose the normalization such that the eigenvalues of the Casimir operator match the conventional eigenvalue formulas [43, 55], given in terms of the Cartan highest-weight labels for the irrep in Table 6. For the symmetric irreps of $SO(n)$ or antisymmetric irreps of $Sp(n)$ arising in the two-level pairing problem, the eigenvalues can be expressed in terms of the single unified formula

$$\left\langle \left\{ \begin{array}{c} C_2[SO(n)] \\ C_2[Sp(n)] \end{array} \right\} \right\rangle = 2v(\theta v + n - 2\theta). \quad (19)$$

Thus, as the second stage of defining the Casimir operator, the normalization is evaluated by explicitly considering the action of the operator on the one-body states, for which the irrep labels are known.

For the *single-level* algebra, $SO(n)$ or $Sp(n)$, the operator $G \circ G$, defined by^{||}

$$G \circ G = - \sum_{g \text{ odd}} \hat{g}[G^{(g)} \times G^{(g)}]_0^{(0)}, \quad (20)$$

commutes with all the generators, *i.e.*, $G^{(g)}$ with g odd [24, 31, 46]. The result follows from the general theory of Casimir operators for an algebra, and it may be verified by explicitly evaluating the commutator $[G^{(g)}, G \circ G]^{(g)}$, using commutator (13) and product rule (A.6). In terms of the conventional spherical tensor scalar product, defined by $A^{(c)} \cdot B^{(c)} = (-)^c \hat{c}(A^{(c)} \times B^{(c)})_0^{(0)}$, this operator is $G \circ G = \sum_{g \text{ odd}} G^{(g)} \cdot G^{(g)}$. The eigenvalue of $G \circ G$ acting on the one-body state $|a\rangle$ may easily be evaluated by Wick's

^{||} The generators of $SO(n)$ or $Sp(n)$ together transform as an $SO(n)$ or $Sp(n)$ tensor. Therefore, the circle in the notation $G \circ G$ is meant to represent a scalar product with respect to $SO(n)$ or $Sp(n)$, following Ref. [56]. In (22) the notation is generalized to represent an $SO_1(n_1) \otimes SO_2(n_2)$ or $Sp_1(n_1) \otimes Sp_2(n_2)$ scalar.

theorem in coupled form [57], using the commutator results described in the appendix. Comparison with the eigenvalue formula (19) gives normalization

$$\left\{ \begin{array}{c} C_2[\text{SO}(n)] \\ C_2[\text{Sp}(n)] \end{array} \right\} = 4G \circ G, \quad (21)$$

covering both the bosonic and fermionic cases.

Proceeding to the *two-level* problem, for the Casimir operator of $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$, we start from the Casimir operators $4G_{aa} \circ G_{aa}$ and $4G_{bb} \circ G_{bb}$ of each single-level subalgebra, as defined in (20). Although each of these operators commutes with each of the generators of $\text{SO}_1(n_1) \otimes \text{SO}_2(n_2)$ or $\text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$, they do not commute with the two-level generators $F^{(g)}$. We therefore introduce the operator

$$F \circ F = \sum_g \hat{g}[F^{(g)} \times F^{(g)}]_0^{(0)}, \quad (22)$$

or, equivalently, $F \circ F = \sum_g (-)^g F^{(g)} \cdot F^{(g)}$, with $g = |j_a - j_b|, \dots, j_a + j_b$. This quantity is invariant with respect to $\text{SO}_1(n_1) \otimes \text{SO}_2(n_2)$ or $\text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$. Moreover, the combination

$$\left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\} = 2\theta F \circ F + 4G_{aa} \circ G_{aa} + 4G_{bb} \circ G_{bb} \quad (23)$$

commutes with all the generators of $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$, as seen by application of the commutators in Table 5 and the product rule (A.6). That this combination of operators also has the correct normalization to match the eigenvalue formula for $C_2[\text{SO}(n)]$ (n even) or $C_2[\text{Sp}(n)]$ (Table 6) may be verified by explicitly calculating the one-body expectation value $\langle a|C_2|a \rangle$ or $\langle b|C_2|b \rangle$.

Returning to the example of the IBM $\text{SO}(6) \supset \text{SO}(5)$ chain, the Casimir operator (21) becomes

$$C_2[\text{SO}(5)] = 4(d^\dagger \times \tilde{d})^{(1)} \cdot (d^\dagger \times \tilde{d})^{(1)} + 4(d^\dagger \times \tilde{d})^{(3)} \cdot (d^\dagger \times \tilde{d})^{(3)}, \quad (24)$$

for the single level consisting of the quadrupole boson $d^{(2)}$. Then, for the $\text{SO}(6)$ algebra of the two-level s - d system,

$$C_2[\text{SO}(6)] = 2[(s^\dagger \times \tilde{d})^{(2)} + (d^\dagger \times \tilde{s})^{(2)}] \cdot [(s^\dagger \times \tilde{d})^{(2)} + (d^\dagger \times \tilde{s})^{(2)}] + C_2[\text{SO}(5)], \quad (25)$$

consistent with the usual result [22].

Similar results apply to the quadratic Casimir operators of the unitary algebras in (5) and (6). For the *single-level* algebra, the linear invariant of $\text{U}(n)$ is simply the occupation number operator $N = \sum_m a_m^\dagger a_m$, or $N = \theta \hat{j} G_0^{(0)}$. The quadratic invariant is given by

$$C_2[\text{U}(n)] = \sum_g \hat{g}(-)^g [G^{(g)} \times G^{(g)}]_0^{(0)}, \quad (26)$$

or $C_2[\text{U}(n)] = \sum_g G^{(g)} \cdot G^{(g)}$. However, for the bosonic realization of $\text{U}(n)$, only symmetric irreps arise, and, for the fermionic realization of $\text{U}(n)$, only antisymmetric irreps arise, with eigenvalues as given in Table 6. Therefore, in either situation, it is

found that the quadratic invariant is simply a function of the linear invariant and can be expressed as

$$C_2[U(n)] = N(\theta N + n - \theta). \quad (27)$$

Likewise, the Casimir operator of the two-level system's algebra $U(n_1 + n_2)$ may be expressed as

$$C_2[U(n_1 + n_2)] = 2\theta N_1 N_2 + n_2 N_1 + n_1 N_2 + C_2[U_1(n_1)] + C_2[U_2(n_2)]. \quad (28)$$

This result is obtained by comparison of the eigenvalues for $C_2[U(n_1 + n_2)]$ with those for $C_2[U(n_1)]$ and $C_2[U(n_2)]$, together with the additivity of the number operators ($N = N_1 + N_2$).

4. Quasispin algebra

First, we note that a set of three operators S_+ , S_- , and S_z obeying the commutation relations

$$[S_0, S_+] = +S_+ \quad [S_0, S_-] = -S_- \quad [S_+, S_-] = -2\theta S_0, \quad (29)$$

and obeying the unitarity conditions $S_+^\dagger = S_-$ and $S_0^\dagger = S_0$, span a unitary realization either of the algebra $SU(1, 1)$, for $\theta = +$, or of the algebra $SU(2)$, for $\theta = -$. The $SU(1, 1)$ or $SU(2)$ invariant operator is given by

$$\begin{aligned} \mathbf{S}^2 &= S_0^2 - \frac{1}{2}\theta(S_+ S_- + S_- S_+) \\ &= S_0(S_0 - 1) - \theta S_+ S_- \end{aligned} \quad (30)$$

For an irrep of $SU(1, 1)$, this operator takes on eigenvalues $S(S - 1)$, and the possible eigenvalues of S_0 are given by $M = S, S + 1, S + 2, \dots$. For the “true” group representations of $SU(1, 1)$, S must be integer or half-integer, but description of the bosonic system in the quasi-spin formalism as considered below requires the projective representations with $S = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$, for reasons described in Ref. [34]. As usual, for $SU(2)$, \mathbf{S}^2 takes on eigenvalues $S(S + 1)$, with $S = 0, \frac{1}{2}, 1, \dots$, and the eigenvalues of S_0 are given by $M = -S, \dots, +S - 1, +S$.

Now, consider a system consisting of one or more j -shells of angular momentum j_k ($k = 1, 2, \dots$). Regardless of whether the operators a_{km}^\dagger for the k th level are bosonic or fermionic, a quasispin algebra is defined following the prescription of Sec. 2. The scalar pair creation operator, scalar pair annihilation operator, and an operator simply related to the number operator for this level form a closed set under commutation. Specifically, let

$$\begin{aligned} S_{k+} &= \frac{1}{2} \sum_m a_{km}^\dagger a_{k\bar{m}}^\dagger \\ S_{k-} &= \frac{1}{2} \sum_m a_{k\bar{m}} a_{km} \\ S_{k0} &= \frac{1}{4} \sum_m (a_{km}^\dagger a_{km} + \theta a_{km} a_{k\bar{m}}^\dagger). \end{aligned} \quad (31)$$

These define either an $SU(1, 1)$ quasispin algebra for bosons [34], which we denote by $SU_k(1, 1)$, or an $SU(2)$ quasispin algebra for fermions [25, 26], which we denote by

$\mathcal{SU}_k(2)$. In terms of spherical tensor coupled products, these operators (31) may be represented as

$$\begin{aligned} S_{k+} &= \frac{1}{2}\hat{j}_k(a_k^\dagger \times a_k^\dagger)_0^{(0)} \\ S_{k-} &= \frac{1}{2}\theta\hat{j}_k(\tilde{a}_k \times \tilde{a}_k)_0^{(0)} \\ S_{k0} &= \frac{1}{4}\hat{j}_k\theta[(a_k^\dagger \times \tilde{a}_k)_0^{(0)} + (\tilde{a}_k \times a_k^\dagger)_0^{(0)}]. \end{aligned} \quad (32)$$

The operator S_{k0} is related to the number operator $N_k = \sum_m a_{km}^\dagger a_{km}$, which may be expressed in spherical tensor form as $N_k = \hat{j}_k(a_k^\dagger \times \tilde{a}_k)_0^{(0)}$, by

$$S_{k0} = \frac{1}{2}(N_k + \theta\Omega_k), \quad (33)$$

where the constant $\Omega_k = \frac{1}{2}(2j_k + 1)$ is the pair degeneracy of the level k . The quasispin S_k , moreover, is related to the seniority quantum number v_k by the duality relation for a single j -shell.

It is therefore possible to interconvert between quasispin quantum numbers S_k and M_k , for each level, and occupation-seniority quantum numbers N_k and v_k , according to

$$\begin{aligned} S_k &= \frac{1}{2}(\Omega_k + \theta v_k) \\ M_k &= \frac{1}{2}(N_k + \theta\Omega_k). \end{aligned} \quad (34)$$

Since the lowest weight state for a given quasispin (*i.e.*, with $M_k = \theta S_k$) is destroyed by the pair annihilation operator, and since this state contains $N_k = v_k$ particles, v_k may be interpreted as the number of unpaired particles, either bosons or fermions. The seniority v_k takes on values $v_k = 0, 1, 2, \dots$, subject to the constraint $v_k \leq N_k$ for bosons or $v_k \leq \min(N_k, 2\Omega_k - N_k)$ for fermions, by the M contents of the irreducible representations noted above.¶ Note, therefore, in the fermionic case, that $v_k \leq \Omega_k$, with the maximum value occurring for half filling ($N_k = \Omega_k$).

A quasispin algebra for the two-level system — which we denote by $\mathcal{SU}_{12}(1, 1)$ for the bosonic case or $\mathcal{SU}_{12}(2)$ for the fermionic case — is spanned by the sum generators

$$S_+ = S_{1+} + \sigma S_{2+} \quad S_- = S_{1-} + \sigma S_{2-} \quad S_0 = S_{10} + S_{20}. \quad (35)$$

A quasispin algebra is obtained with either choice of sign $\sigma = \pm$ in the ladder operators. This algebra defines a total quasispin quantum number S which is dual to the two-level algebra seniority quantum number v , by a relation of the same form as (34), namely,

$$\begin{aligned} S &= \frac{1}{2}(\Omega + \theta v) \\ M &= \frac{1}{2}(N + \theta\Omega), \end{aligned} \quad (36)$$

where N and Ω are defined above as the sums of the single-level values. The allowed values for the total quasispin S are given for $\mathcal{SU}_{12}(1, 1)$ by $S \geq S_1 + S_2$ (*i.e.*,

¶ As noted in Sec. 3.1, the case of a $j = 0$ bosonic level is anomalous. There is no orthogonal algebra dual to the quasispin algebra, and thus no seniority quantum number, but the label v_k may still be defined from the quasispin via (34). The squared quasispin operator is identically $\mathbf{S}^2 = -\frac{3}{16}$ for such a level, as described in Refs. [34, 58]. Thus $S_k = \frac{1}{4}$ or $\frac{3}{4}$, and hence $v_k = 0$ or 1 . Since $M - S$ must be integral, it follows that $v_k = 0$ for N_k even and $v_k = 1$ for N_k odd, *i.e.*, $v_k = N_k \bmod 2$. The natural interpretation of this value is that particles in a $j = 0$ level are automatically paired to zero angular momentum, except for the one unpartnered particle when the occupation is odd.

$S_1 \otimes S_2 \rightarrow S_1 + S_2, S_1 + S_2 + 1, \dots$) and for $\mathcal{SU}_{12}(2)$ by the familiar triangle inequality $|S_1 - S_2| \leq S \leq S_1 + S_2$ (*i.e.*, $S_1 \otimes S_2 \rightarrow |S_1 - S_2|, \dots, S_1 + S_2 - 1, S_1 + S_2$).

When reexpressed in terms of the seniority labels v_1, v_2 , and v , the $\mathcal{SU}(1, 1)$ coupling rule is equivalent to the $\text{SO}(n_1 + n_2) \rightarrow \text{SO}_1(n_1) \otimes \text{SO}_2(n_2)$ branching rule (8), and the $\mathcal{SU}(2)$ coupling rule is equivalent to the $\text{Sp}(n_1 + n_2) \rightarrow \text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2)$ branching rule (10). Similarly, the $\mathcal{SU}(1, 1)$ M content rule is equivalent to the $\text{U}(n) \rightarrow \text{SO}(n)$ branching rule (7), and the $\mathcal{SU}(2)$ M content rule is equivalent to the $\text{U}(n) \rightarrow \text{Sp}(n)$ branching rule (9).

5. Hamiltonian relations

5.1. Dynamical symmetries

Before considering the pairing Hamiltonian in particular, let us consider the Hamiltonian defined by the Casimir operators of the unitary subalgebra chains. A dynamical symmetry [43] arises when the Hamiltonian is constructed in terms of the Casimir operators of a *single* chain of subalgebras. The eigenstates thus reduce the subalgebra chain, *i.e.*, constitute irreps of the subalgebras. More generally, especially when considering phase transitions, it is useful to construct the Hamiltonian from terms consisting of Casimir operators from multiple, parallel chains, here the upper and lower chains of (5) or (6), as

$$H = aN + b_1N_1 + b_2N_2 + b \left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\} \\ + c_1 \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} + c_2 \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\} + d_1\mathbf{J}_1^2 + d_2\mathbf{J}_2^2 + e\mathbf{J}^2, \quad (37)$$

where higher-order invariants may also be included

The upper chain in (5) or (6) defines an $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ dynamical symmetry, and the lower chain defines a $\text{U}_1(n_1) \otimes \text{U}_2(n_2)$ dynamical symmetry. The $\text{U}_1(n_1) \otimes \text{U}_2(n_2)$ dynamical symmetry is obtained for the Hamiltonian (37) with $b = 0$, *i.e.*,

$$H = aN + b_1N_1 + b_2N_2 \\ + c_1 \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} + c_2 \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\} + d_1\mathbf{J}_1^2 + d_2\mathbf{J}_2^2 + e\mathbf{J}^2. \quad (38)$$

The eigenstates $|NN_1N_2v_1v_2 \cdots J_1J_2J\rangle$ have definite occupation numbers for each of the levels and have energy eigenvalues

$$E = aN + b_1N_1 + b_2N_2 + 2c_1v_1(\theta v_1 + n_1 - 2\theta) + 2c_2v_2(\theta v_2 + n_2 - 2\theta) \\ + d_1J_1(J_1 + 1) + d_2J_2(J_2 + 1) + eJ(J + 1). \quad (39)$$

The $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ dynamical symmetry is obtained for the Hamiltonian (37) with $b_1 = b_2 = 0$, *i.e.*,

$$H = aN + b \left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\}$$

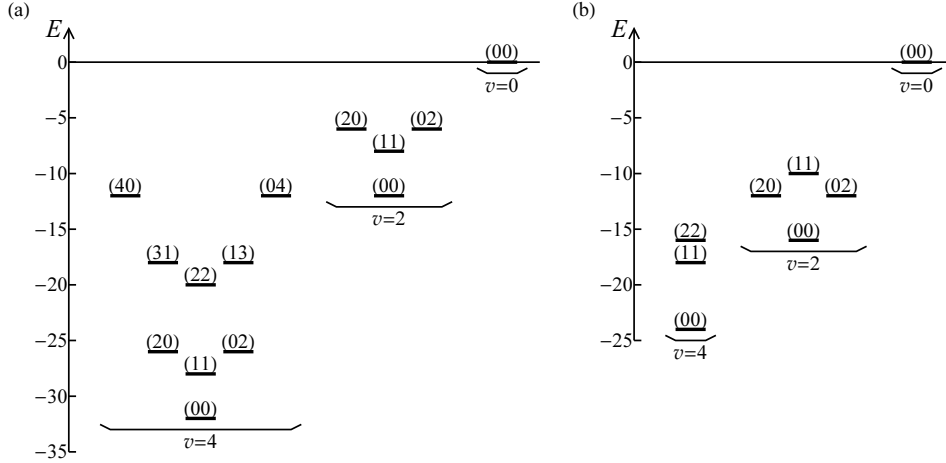


Figure 1. Energy diagrams for the bosonic $\text{SO}(n_1 + n_2)$ and fermionic $\text{Sp}(n_1 + n_2)$ dynamical symmetries of the two-level pairing model. The degeneracy within irreducible representations of the two-level algebra (brackets labeled by v) is split according to the single-level algebra irreducible representation labels $(v_1 v_2)$. (a) Energy levels for the bosonic $\text{SO}(6) \supset \text{SO}(3) \otimes \text{SO}(3)$ dynamical symmetry Hamiltonian ($j_1 = j_2 = 1$). (b) Energy levels for the fermionic $\text{Sp}(8) \supset \text{Sp}(4) \otimes \text{Sp}(4)$ dynamical symmetry Hamiltonian ($j_1 = j_2 = \frac{3}{2}$). The Hamiltonian in each case is chosen as $H = -\theta F \circ F$, i.e., $b = -\frac{1}{2}$ and $c_1 = c_2 = +\frac{1}{2}$ in (40). The total occupation in both cases is $N = 4$, which for the fermionic example gives half filling.

$$+c_1 \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} + c_2 \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\} + d_1 \mathbf{J}_1^2 + d_2 \mathbf{J}_2^2 + e \mathbf{J}^2, \quad (40)$$

with eigenstates $|N v v_1 v_2 \cdots J_1 J_2 J\rangle$ and energy eigenvalues

$$E = aN + 2bv(\theta v + n_1 + n_2 - 2\theta) + 2c_1 v_1(\theta v_1 + n_1 - 2\theta) + 2c_2 v_2(\theta v_2 + n_2 - 2\theta) + d_1 J_1(J_1 + 1) + d_2 J_2(J_2 + 1) + eJ(J + 1). \quad (41)$$

The energy spectrum for the dynamical symmetry follows from the branching rules of Sec. 3.2. Example level energy diagrams are shown for a bosonic system ($n_1 = n_2 = 3$) in Fig. 1(a) and for a fermionic system of similar degeneracies ($n_1 = n_2 = 4$) in Fig. 1(b).

5.2. Pairing Hamiltonian

The pairing Hamiltonian for a generic multi-level system, consisting of levels of angular momentum j_1, j_2, \dots , is given by

$$H = \sum_{km} \varepsilon_k a_{km}^\dagger a_{km} + \frac{1}{4} \sum_{\substack{k'k \\ m'm}} G_{k'k}(-)^{j_{k'}-m'} a_{k'm'}^\dagger a_{k',-m'}^\dagger (-)^{j_k-m} a_{k,-m} a_{km}, \quad (42)$$

where the summation indices k and k' run over the single-particle levels ($k = 1, 2, \dots$), and m and m' run over their substates ($m = -j_k, -j_k + 1, \dots, +j_k$). The creation operators a_{km}^\dagger and annihilation operators a_{km} obey either canonical commutation

relations (bosons) or anticommutation relations (fermions). The first term represents the one-body energy contribution for each level, and the products in the second term are creation or annihilation operators for pairs involving time-reversed partner substates. Note that $G_{k'k} = G_{kk'}^*$ by Hermiticity of H , and that we take all coefficients to be real. That this Hamiltonian is specifically constructed from angular momentum zero pair operators is seen by rewriting it in terms of angular-momentum coupled products, as

$$H = \sum_k \varepsilon_k N_k + \frac{1}{4}\theta \sum_{k'k} G_{k'k} \hat{j}_{k'} \hat{j}_k (a_{k'}^\dagger \times a_k^\dagger)_0^{(0)} (\tilde{a}_k \times \tilde{a}_{k'})_0^{(0)}. \quad (43)$$

The Hamiltonian (42) is integrable and may be solved using a generalized Gaudin algebra [59] or, under certain conditions, Bethe ansatz [60, 61] methods. Two limiting cases deserve special mention, since they are characterized by dynamical symmetries (Sec. 5.1), as described below, and are solvable by more elementary methods. (1) Trivially, when all $G_{k'k} = 0$, the problem reduces to that of a system of noninteracting particles (weak-coupling limit), with eigenstates characterized by occupations numbers N_k . (2) When all $\varepsilon_k = 0$, physically corresponding to the situation in which the level energy difference is negligible relative to the pairing strength (strong-coupling limit), and if all $G_{k'k}$ are equal to within possible phase factors (uniform pairing), the problem is immediately solvable by the use of quasispin (Sec. 5.3).

5.3. Quasispin Hamiltonian

The generic multi-level pairing Hamiltonian (42) can be expressed entirely in terms of the quasispin generators, by comparison with (32), as

$$H = \sum_k \varepsilon_k (2S_{k0} - \theta\Omega_k) + \sum_{k'k} G_{k'k} S_{k'+} S_{k-}. \quad (44)$$

This Hamiltonian therefore conserves the quasispin S_k or, equivalently, the seniority v_k , associated with each level. The Hamiltonian is also number conserving, so the total z -projection $S_0 = \sum_k S_{k0} = \frac{1}{2}(N + \theta\Omega)$ is conserved, where $N = \sum_k N_k$ and $\Omega = \sum_k \Omega_k$. However, the individual S_{k0} are not in general conserved, unless the levels completely decouple, with $G_{k'k} = 0$ for all $k' \neq k$.

Note that numerical diagonalization is straightforward in the weak-coupling basis, consisting of states $|N_1 v_1 N_2 v_2 \dots\rangle$ of good seniority and occupation for each level. The action of the Hamiltonian (44) on these states follows from the known action of the quasispin ladder operators, $S_\pm |SM\rangle = [-\theta(S \pm \theta M)(S \mp \theta M - \theta)]^{1/2} |S(M \pm 1)\rangle$, once the quantum numbers are translated via (34). Specifically,

$$\begin{aligned} S_{k+} |\dots N_k v_k \dots\rangle &= \frac{1}{2} [\theta(N_k - v_k + 2)(N_k + v_k + 2\theta\Omega_k)]^{1/2} |\dots (N_k + 2) v_k \dots\rangle \\ S_{k-} |\dots N_k v_k \dots\rangle &= \frac{1}{2} [\theta(N_k - v_k)(N_k + v_k + 2\theta\Omega_k - 2)]^{1/2} |\dots (N_k - 2) v_k \dots\rangle. \end{aligned} \quad (45)$$

Consequently, the matrix elements for the diagonal pairing terms are

$$\langle \dots N_k v_k \dots | S_{k+} S_{k-} | \dots N_k v_k \dots \rangle = \frac{1}{4} \theta [N_k(N_k + 2\theta\Omega_k - 2) - v_k(v_k + 2\theta\Omega_k - 2)] \quad (46)$$

and for the off-diagonal terms are

$$\begin{aligned} \langle \cdots (N_{k'} + 2)v_{k'} \cdots (N_k - 2)v_k \cdots | S_{k'+} S_{k-} | \cdots N_{k'} v_{k'} \cdots N_k v_k \cdots \rangle \\ = \frac{1}{4} [(N_{k'} - v_{k'} + 2)(N_{k'} + v_{k'} + 2\theta\Omega_{k'}) \\ \times (N_k - v_k)(N_k + v_k + 2\theta\Omega_k - 2)]^{1/2}, \end{aligned} \quad (47)$$

as noted for the fermionic case in, *e.g.*, Refs. [10, 40].

Returning to the two-level problem, the pairing Hamiltonian in quasispin notation is

$$\begin{aligned} H = \varepsilon_1(2S_{10} - \theta\Omega_1) + G_{11}S_{1+}S_{1-} + \varepsilon_2(2S_{20} - \theta\Omega_2) + G_{22}S_{2+}S_{2-} \\ + G_{12}(S_{1+}S_{2-} + S_{2+}S_{1-}). \end{aligned} \quad (48)$$

The two-level pairing Hamiltonian has two dynamical symmetries [62] defined with respect to the quasispin algebras, corresponding to either the upper or lower subalgebra chains in

$$\mathcal{SU}_{v_1}(1, 1) \otimes \mathcal{SU}_{v_2}(1, 1) \supset \left\{ \begin{array}{c} \mathcal{SU}_{12}(1, 1) \\ v \\ \mathcal{U}_1(1) \otimes \mathcal{U}_2(1) \\ N_1 \quad N_2 \end{array} \right\} \supset \mathcal{U}_{12}(1)_N \quad (49)$$

for the bosonic case or

$$\mathcal{SU}_{v_1}(2) \otimes \mathcal{SU}_{v_2}(2) \supset \left\{ \begin{array}{c} \mathcal{SU}_{12}(2) \\ v \\ \mathcal{U}_1(1) \otimes \mathcal{U}_2(1) \\ N_1 \quad N_2 \end{array} \right\} \supset \mathcal{U}_{12}(1)_N \quad (50)$$

for the fermionic case, with conserved quantum numbers as indicated. Here the algebra $\mathcal{U}_k(1)$ is the trivial Abelian algebra spanned by S_{k0} , and $\mathcal{U}_{12}(1)$ is spanned by the their sum S_0 , as defined in (35). The occupation-seniority labels (v and N) are indicated, rather than the quasispin labels (S and M), for a closer connection to the physical problem and easier comparison with the dual algebra's dynamical symmetries.

The dynamical symmetry Hamiltonian for the upper subalgebra chain is the strong coupling limit (*i.e.*, $\varepsilon_1 = \varepsilon_2 = 0$) of the two-level pairing Hamiltonian, with uniform pairing strength, as defined in Sec. 5.2. Specifically, let $G_{11} = \sigma G_{12} = \sigma G_{21} = G_{22} \equiv G$, for either sign $\sigma = \pm$. Then the Hamiltonian is given by

$$H = GS_+S_-, \quad (51)$$

where S_{\pm} are the sum-quasispin ladder operators of (35), defined in terms of the same sign σ . Since $S_+S_- = \theta[S_0(S_0 - 1) - \mathbf{S}^2]$, by (30), the strong-coupling Hamiltonian conserves the total quasispin S (or seniority v), as well as the projection quantum number M (or occupation N), and has eigenvalues

$$\langle S_+S_- \rangle = \frac{1}{4}\theta[N(N + 2\theta\Omega - 2) - v(v + 2\theta\Omega - 2)], \quad (52)$$

as expressed in terms of the occupation-seniority labels. The eigenstates are identical to those of the $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ dynamical symmetry Hamiltonian (40).⁺ The

⁺ More precisely, the quasispin Hamiltonian (51) has a higher degeneracy than the $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ Hamiltonian (40), but the eigenstates can be chosen from within each degenerate subspace to match those of (40), *i.e.*, of good J_1 , J_2 , and J .

specific relationship between the Hamiltonians is determined below in Sec. 5.4.

The dynamical symmetry Hamiltonian for the lower subalgebra chain is the weak-coupling limit of the pairing Hamiltonian ($G_{k'k} = 0$), as defined in Sec. 5.2. The dynamical symmetry eigenstates are simply the level occupation eigenstates of good N_1 and N_2 , as for the $U_1(n_1) \otimes U_2(n_2)$ dynamical symmetry Hamiltonian (38), *i.e.*, the weak-coupling basis states considered above.

The full two-level pairing Hamiltonian (48) can be expressed entirely in terms of the invariant operators of algebras appearing in the upper and lower chains. Specifically,

$$\begin{aligned} H = & \varepsilon_1(2S_{10} - \theta\Omega_1) + (G_{11} - \sigma G_{12})\theta[S_{10}(S_{10} - 1) - \mathbf{S}_1^2] \\ & + \varepsilon_2(2S_{20} - \theta\Omega_2) + (G_{22} - \sigma G_{12})\theta[S_{20}(S_{20} - 1) - \mathbf{S}_2^2] \\ & + G_{12}\sigma\theta[S_0(S_0 - 1) - \mathbf{S}^2]. \end{aligned} \quad (53)$$

5.4. Duality relations for the Hamiltonian

The eigenstates for the dynamical symmetries of the two algebraic frameworks — number-conserving unitary and number-nonconserving quasispin — are identical, that is, the irreps which reduce the unitary algebra chains (5) and (6) reduce the quasispin algebra chains (49) and (50) as well, and the labels for the chains are connected through the duality relations. The pairing Hamiltonian is defined in Sec. 5.2 [see (42)] in terms of certain combinations of operators which represent scalar pair creation, scalar pair annihilation, and number operators. These are noted in Sec. 5.3 [see (44)] to be essentially the quasispin generators, and the Hamiltonian can also therefore be expressed directly in terms of the quasispin invariants [see (53)]. However, the pairing Hamiltonian can just as well be expressed in terms of the Casimir operators of subalgebras of $U(n_1+n_2)$, as a special case of the Hamiltonian of Sec. 5.1 [see (37)]. That such a relation exists is implied by the duality of irreps, but it is explicitly obtained by appropriate recoupling and reordering of the bosonic or fermionic creation and annihilation operators in this section.

For a *single* j -shell, recoupling and commutation of creation operators* yields

$$\begin{aligned} (a^\dagger \times a^\dagger)_0^{(0)} (\tilde{a} \times \tilde{a})_0^{(0)} \\ = \frac{\theta}{j^2} \left[-(a^\dagger \times \tilde{a})_0^{(0)} + \sum_g \hat{g} [(a^\dagger \times \tilde{a})^{(g)} \times (a^\dagger \times \tilde{a})^{(g)}]_0^{(0)} \right]. \end{aligned} \quad (54)$$

Thus, the relation between quasispin and Casimir Hamiltonians for a single level is

$$4S_+S_- = -\theta N + C_2[U(n)] - \frac{1}{2} \left\{ \begin{array}{c} C_2[\text{SO}(n)] \\ C_2[\text{Sp}(n)] \end{array} \right\}, \quad (55)$$

by comparison with the explicit realizations of the various operators, namely, N from Sec. 3.4, $C_2[\text{SO}(n)]$ or $C_2[\text{Sp}(n)]$ from (21), $C_2[U(n)]$ from (26), and S_\pm from (32).

* The product of a pair creation operator and a pair annihilation operator is related to the product of spherical-tensor one-body operators by, *e.g.*, identity (25a) of Ref. [57].

For the *two-level* system, which has quasispin generators given by (35), the product S_+S_- involves both single-level terms ($S_{1+}S_{1-}$ and $S_{2+}S_{2-}$) and “cross terms” ($\sigma S_{1+}S_{2-}$ and $\sigma S_{2+}S_{1-}$), which destroy a pair in one level and create a pair in the other. Recoupling and commutation of the mixed terms yields

$$\begin{aligned} (a^\dagger \times a^\dagger)_0^{(0)} (\tilde{b} \times \tilde{b})_0^{(0)} + (b^\dagger \times b^\dagger)_0^{(0)} (\tilde{a} \times \tilde{a})_0^{(0)} \\ = \frac{(-)^{j_a+j_b}\theta\sigma_0}{\hat{j}_a\hat{j}_b} [F \circ F - 2N_aN_b - \theta(\hat{j}_b^2N_a + \hat{j}_a^2N_b)]. \end{aligned} \quad (56)$$

The one-level terms and mixed terms of S_+S_- may thus be combined to give an expression involving the $\text{SO}(n_1 + n_2)$ or $\text{Sp}(n_1 + n_2)$ Casimir operator and $\text{U}(n_1 + n_2)$ invariants, *if and only if* the sign σ arising in the definition of the sum quasispin algebra and the sign σ_0 entering into the definition of $F^{(g)}$ are related by

$$\frac{\sigma_0}{\sigma} = -\theta(-)^{j_a+j_b}. \quad (57)$$

We again have an expression of the same form as (55),

$$4S_+S_- = -\theta(N_1 + N_2) + C_2[\text{U}(n_1 + n_2)] - \frac{1}{2} \left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\}. \quad (58)$$

That the expression is of this form is to be expected from the general nature of the duality, which indeed makes no assumption (Sec. 2) as to whether the single-particle states are considered to be arranged into a single j -shell, as for (55), or two j -shells, as here. From a practical standpoint, what is most useful is that the relation can now be expressed explicitly in terms of the two-level system operators given by (23), (28), and (35) as spherical-tensor products of creation and annihilation operators, with well-defined phases σ_0 and σ .

The two-level operator correspondence (58) relates the $\mathcal{SU}_{12}(1,1)$ or $\mathcal{SU}_{12}(2)$ dynamical symmetry Hamiltonian in the quasispin scheme (51), *i.e.*, strong coupling with uniform pairing, to the $\text{SO}(n_1+n_2)$ or $\text{Sp}(n_1+n_2)$ dynamical symmetry Hamiltonian in the unitary algebra scheme.[‡] Furthermore, taken in conjunction with the single-level operator correspondence (55), it allows the full two-level pairing Hamiltonian (not just at the dynamical symmetry limit) to be expressed in terms of Casimir operators of algebras appearing in the two parallel subalgebra chains of $\text{U}(n_1 + n_2)$. Starting from (48), one obtains

$$\begin{aligned} H = (\varepsilon_1 - \frac{1}{4}\theta G_{11})N_1 + \frac{1}{4}(G_{11} - \sigma G_{12})C_2[\text{U}_1(n_1)] - \frac{1}{8}(G_{11} - \sigma G_{12}) \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} \\ + (\varepsilon_2 - \frac{1}{4}\theta G_{22})N_2 + \frac{1}{4}(G_{22} - \sigma G_{12})C_2[\text{U}_2(n_2)] - \frac{1}{8}(G_{22} - \sigma G_{12}) \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\} \\ + \frac{1}{4}\sigma G_{12}C_2[\text{U}(n_1 + n_2)] - \frac{1}{8}\sigma G_{12} \left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\}. \end{aligned} \quad (59)$$

[‡] The analog of (58) for the IBM was exploited in Ref. [38] to establish the properties of the IBM $\text{SO}(6)$ dynamical symmetry eigenstates. With $\theta = +$, the sign condition (57) gives $\sigma_0 = -\sigma$ (recall $j_a = 0$, and $j_b = 2$). Thus, the pairing operator S_+S_- for the quasispin defined with *negative* sign ($\sigma = -$) relates to the Casimir operator of the “physical” $\text{SO}(6)$ algebra ($\sigma_0 = +$), and the quasispin algebra defined with *positive* relative sign is instead dual to the $\overline{\text{SO}}(6)$ algebra (Sec. 3.3).

For this relation to be valid, the phases σ_0 and σ , used in defining the generators for orthogonal or symplectic algebra and two-level quasispin algebra, respectively, must be related by (57).

5.5. Multipole Hamiltonian

The Hamiltonian for spectroscopic studies of the s - b boson models is commonly expressed in terms of a “multipole” term of the form $[(s^\dagger \times \tilde{b})^{(L)} + (b^\dagger \times \tilde{s})^{(L)}] \cdot [(s^\dagger \times \tilde{b})^{(L)} + (b^\dagger \times \tilde{s})^{(L)}]$, where $L = j_b$ [63]. For instance, the customary IBM U(5)–SO(6) quadrupole Hamiltonian [64, 65] is

$$H_{QQ} = \frac{(1-\xi)}{N} N_d - \frac{\xi}{N^2} [(s^\dagger \times \tilde{d})^{(2)} + (d^\dagger \times \tilde{s})^{(2)}] \cdot [(s^\dagger \times \tilde{d})^{(2)} + (d^\dagger \times \tilde{s})^{(2)}]. \quad (60)$$

The U(5) limit is obtained for $\xi = 0$ and the SO(6) limit for $\xi = 1$. The operator $F \circ F$, appearing as the “cross term” in $C_2[\text{SO}(n_1 + n_2)]$ or $C_2[\text{Sp}(n_1 + n_2)]$ [see (23)], generalizes the multipole term to the generic two-level model. In contrast, the Hamiltonian for spectroscopic studies of the fermionic system is commonly expressed in pairing or quasispin form [5, 20]. Thus, we seek to relate these distinct — pairing and multipole — forms of the Hamiltonian.

Recall that the operators considered thus far in connection with the strong-coupling limit — \mathbf{S}^2 , $S_+ S_-$, and $C_2[\text{SO}(n_1 + n_2)]$ or $C_2[\text{Sp}(n_1 + n_2)]$ — differ only in normalization (or sign) and by addition of a function of N , the conserved total occupation number. Therefore, the eigenstates are identical and the eigenvalues differ only by a rescaling and a constant offset. However, the operator $F \circ F$ appearing in the multipole Hamiltonian differs from these [again, see (23)] by terms proportional to $C_2[\text{SO}_1(n_1)]$ and $C_2[\text{SO}_2(n_2)]$, in the bosonic case, or $C_2[\text{Sp}_1(n_1)]$ and $C_2[\text{Sp}_2(n_2)]$, in the fermionic case. The *eigenstates* are therefore again the same as for \mathbf{S}^2 , $S_+ S_-$, and $C_2[\text{SO}(n_1 + n_2)]$ or $C_2[\text{Sp}(n_1 + n_2)]$, but *eigenvalues* are no longer degenerate for states sharing the same value of v . They are rather now split by v_1 and v_2 , as illustrated in Fig. 1.

For the explicit relationships among these operators, observe that, by (58),

$$4S_+ S_- = C_2[\text{U}(n_1 + n_2)] - \theta N - \frac{1}{2} \left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\}, \quad (61)$$

and, in terms of (23),

$$\left\{ \begin{array}{c} C_2[\text{SO}(n_1 + n_2)] \\ C_2[\text{Sp}(n_1 + n_2)] \end{array} \right\} = 2\theta F \circ F + \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} + \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\}. \quad (62)$$

Therefore, the pairing ($S_+ S_-$) and multipole ($F \circ F$) forms of the Hamiltonian, those most frequently encountered in applications, are related by

$$\begin{aligned} -\theta F \circ F = 4S_+ S_- - & \left[C_2[\text{U}(n_1 + n_2)] - \theta N \right] \\ & + \frac{1}{2} \left\{ \begin{array}{c} C_2[\text{SO}_1(n_1)] \\ C_2[\text{Sp}_1(n_1)] \end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c} C_2[\text{SO}_2(n_2)] \\ C_2[\text{Sp}_2(n_2)] \end{array} \right\}, \end{aligned} \quad (63)$$

where

$$C_2[U(n_1 + n_2)] - \theta N = \begin{cases} N(N + 2\Omega - 2) \\ N(2\Omega - N + 2) \end{cases} \quad (64)$$

simply contributes a c -number shift to the eigenvalue spectrum, without affecting the eigenfunctions.

A positive coefficient ($G > 0$) for S_+S_- gives a positive pair energy, *i.e.*, repulsive pairing, in both bosonic and fermionic cases, as may be seen from (52) with $N = 1$ and $v = 0$. The sign of the pairing interaction is of special interest in comparing the bosonic and fermionic two-level pairing models, since it should be noted (Sec. 6) that the system undergoes a quantum phase transition for *repulsive* ($G > 0$) pairing interaction in the bosonic case and *attractive* pairing interaction ($G < 0$) in the fermionic case. Thus, it is essential to note that repulsive pairing is obtained for a negative coefficient on $F \circ F$ in the bosonic case and a positive coefficient on $F \circ F$ in the fermionic case, *i.e.*, for $H = -\theta F \circ F$, or *vice versa* for attractive pairing. Therefore, for repulsive pairing, a Hamiltonian

$$H_{FF} = \frac{(1 - \xi)}{N} N_2 - \theta \frac{\xi}{N^2} F \circ F \quad (65)$$

is the natural generalization of the multipole form for the transitional Hamiltonian (60) to generic two-level pairing models, as considered in Sec. 6.

The last two in terms in (63), involving the Casimir operators of $SO_1(n_1)$ and $SO_2(n_2)$ or $Sp_1(n_1)$ and $Sp_2(n_2)$, contribute a common shift to the energy eigenvalues for each subspace of states characterized by a given pair of values of the conserved $(v_1 v_2)$ quantum numbers, without affecting the eigenfunctions, *i.e.*, these terms serve only to displace the different $(v_1 v_2)$ subspaces relative to each other. If only $(v_1 v_2) = (00)$ states are considered, the $SO_1(n_1)$ and $SO_2(n_2)$ or $Sp_1(n_1)$ and $Sp_2(n_2)$ terms have no effect at all. They will therefore not be considered further.

Now to consider the spectra, the strong-coupling Hamiltonian operator $4S_+S_-$ — we include the factor of 4 arising in (63) for convenience — has eigenvalues given by (52), obtained with $0 \leq v \leq N$ for bosonic pairing or with $0 \leq v \leq \min(N, 2\Omega - N)$ for fermionic pairing, where only even values of v arise for N even, or odd values of v for N odd [see (7) and (9)]. Thus, taking N even, the eigenvalues span the range

$$\langle 4S_+S_- \rangle = \underbrace{0}_{v=N}, \dots, \underbrace{N(N + 2\Omega - 2)}_{v=0}, \quad (66)$$

for bosonic pairing, or

$$\langle 4S_+S_- \rangle = \underbrace{0}_{\substack{v=N \\ (N \leq \Omega)}} \quad \text{or} \quad \underbrace{4(N - \Omega)}_{\substack{v=2\Omega-N \\ (N \geq \Omega)}}, \dots, \underbrace{N(2\Omega - N + 2)}_{v=0}, \quad (67)$$

for fermionic pairing, in which case $0 \leq N \leq 2\Omega$. (If N is odd, the sequences above would end instead with $v = 1$ rather than $v = 0$, but the large- N dependence of the highest eigenvalue on N^2 is not changed.) The range of eigenvalues therefore depends upon the total occupation or “filling” N of the two-level system as sketched in Fig. 2(a) for the

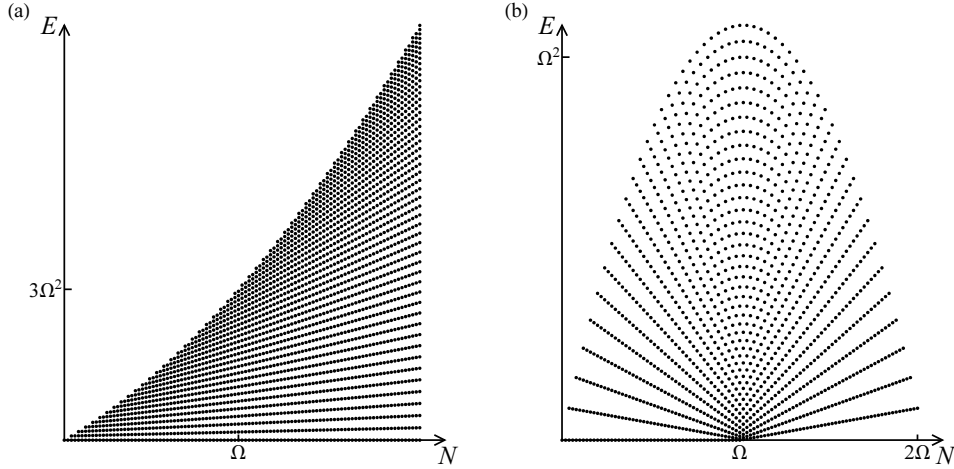


Figure 2. Eigenvalues of the pairing interaction term $4S_+S_-$, which determines the energy spectrum of the two-level pairing model in the strong coupling limit, shown for the (a) bosonic and (b) fermionic systems, as a function of filling N . The eigenvalues are given by (66) and (67), respectively. The axes are labeled generically, to indicate the asymptotic (large- Ω) dependences discussed in the text, but the specific points shown for illustration are calculated for $\Omega = 50$.

bosonic system and Fig. 2(b) for the fermionic system. The asymptotic dependences for large degeneracy ($\Omega \gg 1$) are indicated. Specifically, at a filling approximately equal to half the total degeneracy, *i.e.*, $N \approx \Omega$, note that the bosonic eigenvalues span a range $\sim 3\Omega^2 (\approx 3N^2)$, while the fermionic eigenvalues for the same filling and degeneracy only span a range of $\sim \Omega^2 (\approx N^2)$. If, instead, the limit of large occupation is taken at fixed degeneracy ($\Omega \ll N$) in the bosonic case, the range of eigenvalues $\sim N^2$ is the same as for a fermionic pairing model of the same N but at half filling (which is obtained for a correspondingly larger degeneracy $\Omega = N$).

Note also that, for repulsive pairing, the bosonic ground state (for which $v = N$) has zero eigenvalue. In contrast, the fermionic ground state (for which $v = 0$ below half filling and $v = 2\Omega - N$ past half filling) has an eigenvalue which grows linearly with N past half filling. The nonzero ground state pairing energy for the fermionic system may be understood since, past half filling, Pauli exclusion enforces the existence of some particles in time-reversal conjugate orbits and hence some probability for pairs coupled to zero angular momentum.

For the multipole form $-\theta F \circ F$ of the pairing interaction operator, the spectrum is shifted downward by an N -dependent offset relative to that of $4S_+S_-$ [see (63)]. The *highest* eigenvalue (obtained for $v = 0$) is always zero. The asymptotic form of the ground state eigenvalue is, alternatively, $\sim -N^2$ for fermionic half filling ($1 \ll \Omega = N$), $\sim -3N^2$ for bosonic “half filling” ($1 \ll \Omega \approx N$), and again $\sim -N^2$ for the bosonic system at larger boson number ($\Omega \ll N$).

6. Transitional Hamiltonian

A second-order quantum state phase transition occurs between the weak-coupling and strong-coupling limits for the two-level pairing models. Specifically, for the *bosonic* system it occurs with *repulsive* pairing interaction, and for the *fermionic* system it occurs with *attractive* interaction. The quantum phase transition is apparent numerically from calculations for finite N and from semiclassical treatments of the large- N limit. The present duality relations (Sec. 5) immediately help clarify comparison of numerical eigenvalue spectra across the transition but are also intended to facilitate the construction of coherent states for the semiclassical treatment.

The simplest semiclassical “geometry” for the two-level pairing model is obtained from the quasispin algebraic structure, most simply by replacing the quasispin operators with classical angular momentum vectors, which maps the pairing model onto an essentially one-dimensional coordinate space. This approach has been applied in both the bosonic s - b models and fermionic two-level pairing model with equal degeneracies [5, 8, 42, 66, 67]. In both these circumstances, the quantum phase transition is found to occur, in the large- N limit, at $N|G|/\varepsilon = 1$. For the s - b models, a higher-dimensional and richer classical geometry (see Ref. [42]) has been established through the use of $U(n_2 + 1)/U(n_2)$ coherent states [42, 68–71]. An extension of this treatment to $U(n_1 + n_2)/[U(n_1) \otimes U(n_2)]$ for generic two-level pairing models might profitably be obtained using the explicit construction of generators for the $SO(n_1 + n_2) \supset SO(n_1) \otimes SO(n_2)$ and $Sp(n_1 + n_2) \supset Sp(n_1) \otimes Sp(n_2)$ chains considered in Sec. 3.

However, at present, we confine ourselves to laying the groundwork for more detailed further work, allowing for the most general choice of level degeneracies and more uniformly treating the bosonic and fermionic cases. A pairing Hamiltonian

$$H_{\text{pair}} = \frac{(1 - \xi)}{N} N_2 + \theta \frac{4\xi}{N^2} S_+ S_-, \quad (68)$$

may be defined with opposite signs θ of the pairing term for the bosonic and fermionic cases, so that the quantum phase transition is obtained in either case. This Hamiltonian yields the weak-coupling limit for $\xi = 0$, the strong-coupling limit at $\xi = 1$, and the critical interaction strength $N|G|/\varepsilon = 1$ at $\xi = 1/5$. Scaling of the one-body term by N^{-1} and of the two-body term by N^{-2} ensures that the critical point remains fixed at the finite value $\xi = 1/5$ as $N \rightarrow \infty$. However, for this Hamiltonian, a grossly different “envelope” to the eigenvalue spectrum (*i.e.*, the range of eigenvalues, obtained as a function of the control parameter ξ) is found in the bosonic and fermionic cases (see Fig. 3).

To facilitate direct comparison of the bosonic and fermionic quantum phase transitions, it is helpful to instead construct a Hamiltonian for which the ground state energy follows the same trajectory as a function of ξ in the large N limit, and the eigenvalues span the same range at each of the limits, namely, $[0, 1]$ for $\xi = 0$ and $[-1, 0]$ for $\xi = 1$. By the results of Sec. 5, this is accomplished by choosing, for *repulsive* pairing

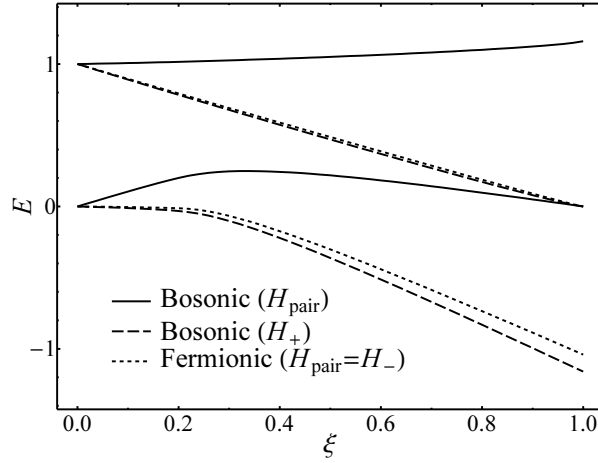


Figure 3. Lowest and highest eigenvalues, within the $(v_1 v_2) = (00)$ subspace, for the bosonic two-level pairing model with repulsive pairing interaction, in the usual form as given by H_{pair} in (68) (solid curves) and after correction by the $U(n_1 + n_2)$ Casimir offset as given by H_+ in (69) (dashed curves). These are plotted as functions of the control parameter ξ between the weak-coupling and strong-coupling limits. The eigenvalues for the fermionic two-level pairing model with attractive pairing interaction, as given by H_{pair} or equivalently by H_- in (70), are shown for comparison (dotted curves). All calculations are for $N = 50$, with $n_1 = n_2 = 5$ in the bosonic case and $n_1 = n_2 = 50$ in the fermionic case (see Fig. 4 for more detailed eigenvalue spectra).

interaction, the Hamiltonian

$$H_+ = \frac{(1 - \xi)}{N} N_2 + \frac{\xi}{N^2} \left[4S_+ S_- - \left\{ \begin{matrix} N(N + 2\Omega - 2) \\ N(2\Omega - N + 2) \end{matrix} \right\} \right], \quad (69)$$

and, for *attractive* pairing interaction, the usual Hamiltonian

$$H_- = \frac{(1 - \xi)}{N} N_2 - \frac{4\xi}{N^2} S_+ S_-. \quad (70)$$

The c -number offset included in the definition of H_+ , which arises as $C_2[U(n_1 + n_2)] - \theta N$, is included to achieve the same range of eigenvalues in the strong coupling limit, as well as a similar evolution of ground state energy across the transition (Fig. 3), in the large- N limit, thereby facilitating comparison of the bosonic (repulsive pairing) and fermionic (attractive pairing) quantum phase transition. With inclusion of this offset, H_+ is equivalent to the generalized multipole transitional Hamiltonian (65) when acting on the $(v_1 v_2) = (00)$ subspace of any two-level pairing model, as may be seen from (63). In particular, for the s - b models, inclusion of this offset makes H_+ identical to the conventional multipole form (60) of the transitional Hamiltonian, when acting on the $(v_1 v_2) = (00)$ subspace.

The evolution of the eigenvalue spectrum across the transition between weak coupling and strong coupling is shown for representative bosonic and fermionic cases in Fig. 4. Specifically, equal-degeneracy pairing models ($n_1 = n_2 \equiv \Omega$) are considered, and the $(v_1 v_2) = (00)$ states are shown. Here a sufficiently large total occupancy

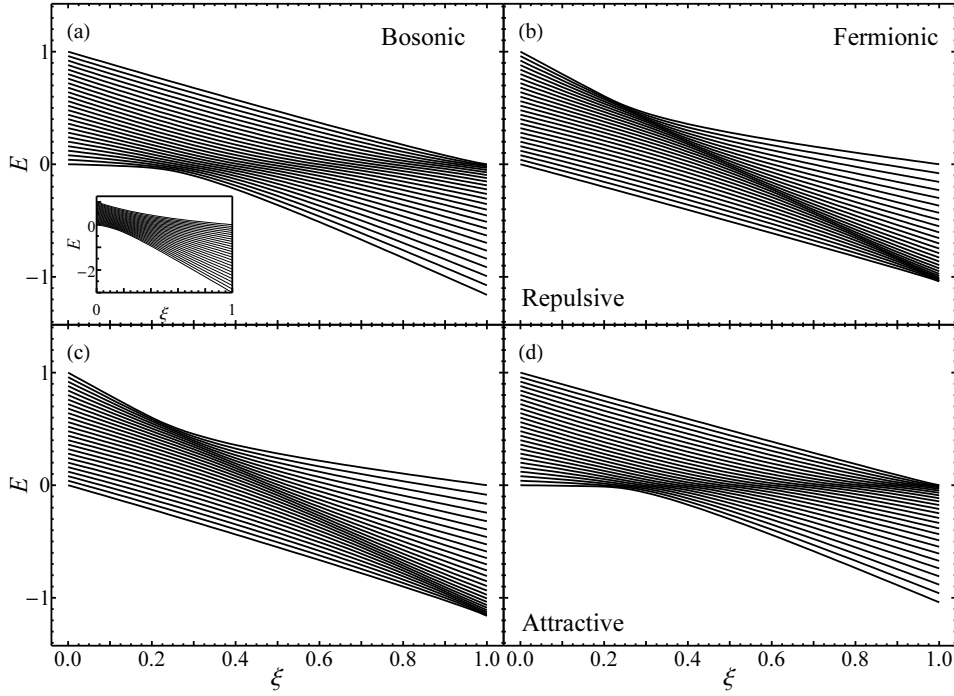


Figure 4. Eigenvalues of the bosonic two-level pairing model, with level degeneracies $n_1 = n_2 = 5$ (at left), and fermionic two-level pairing model, with level degeneracies $n_1 = n_2 = 50$ (at right), for repulsive (at top) and attractive (at bottom) pairing interactions, shown for the $(v_1 v_2) = (00)$ subspace, as functions of the control parameter ξ between the weak-coupling and strong-coupling limits. All calculations are for $N = 50$, thus for $\Omega \ll N$ in the bosonic case and $\Omega = N$ (half filling) in the fermionic case. The alternative regime in which the bosonic system also has $\Omega \approx N$ is shown (specifically, for $n_1 = n_2 = 51$ and $N = 50$) in the inset to panel (a). The Hamiltonians H_{\pm} of (69) and (70) are used in the calculations.

($N = 50$) is chosen such that the precursors of the phase transitional singularities are readily apparent. Spectra are shown for both bosons [Fig. 4 (left)] and fermions [Fig. 4 (right)], with repulsive [Fig. 4 (top)] and attractive [Fig. 4 (bottom)] interactions. Qualitatively similar spectra in the bosonic and fermionic cases are obtained when level degeneracies for the bosonic calculation ($\Omega = 5$) are *much less than* the occupancy, while the degeneracies for the fermionic calculation ($\Omega = 50$) are such as to give *half filling*. Then the “envelope” of the spectrum (the range of eigenvalues at a given value of the Hamiltonian parameter ξ) is essentially identical for the *bosonic* case with *repulsive* pairing [Fig. 4 (a)] and the *fermionic* case with *attractive* pairing [Fig. 4 (d)], *i.e.*, the interactions signs which yield a ground state quantum phase transition. Features to observe include the essentially constant ground state energy for $\xi < 1/5$ and downturn [from 0 to -1] for $\xi > 1/5$, an approximately linear evolution of the highest eigenvalue from $+1$ to 0 , and a compression of the level density at $E \approx 0$ for $\xi > 1/5$, a characteristic of the excited state quantum phase transition [14].

The structure of the eigenvalue spectrum is likewise similar when one compares the

bosonic case with *attractive* pairing [Fig. 4 (c)] and the *fermionic* case with *repulsive* pairing [Fig. 4 (b)]. This should hardly be surprising. Indeed, when $n_1 = n_2$, the eigenvalue spectra for Hamiltonians for opposite pairing signs [*e.g.*, Fig. 4 (a) and Fig. 4 (c), or Fig. 4 (b) and Fig. 4 (d)] may be obtained from each other, by negation of the Hamiltonian and interchange of the level labels 1 and 2, to within addition of a *c*-number function of ξ . Therefore, in the present example, the resemblance between Fig. 4 (c) and Fig. 4 (b) is a necessary consequence of the resemblance between Fig. 4 (a) and Fig. 4 (d).

The emergence of finite-size precursors to the infinite- N singularities associated with the quantum phase transition depends not only on N but also on the level degeneracies n_1 and n_2 . An important distinction therefore arises between bosonic and fermionic models [14]. For *fermionic* systems, the total occupancy N is limited to $n_1 + n_2$. Therefore, the limit of large N can only be taken if the level degeneracies are simultaneously increased. Since at full filling ($N = n_1 + n_2$) the spectrum, like that for zero filling, is trivial, it is more informative to take the limit $N \rightarrow \infty$ at or near half filling [$N = \frac{1}{2}(n_1 + n_2) \equiv \Omega$]. However, no such restriction arises for *bosonic* systems, and $N \rightarrow \infty$ can be obtained even for fixed level degeneracies.

Indeed, for the bosonic two-level pairing models, we find numerically that the onset of critical phenomena requires $N \gg \Omega$, not $N \approx \Omega$. The evolution of eigenvalues for the bosonic system with the same occupancy ($N = 50$) as in Fig. 4(a), but with level degeneracies comparable to the occupation $n_1 = n_2 (= \Omega) = 49 \approx 50$, analogous to “half filling”, is shown for comparison in Fig. 4 (inset). The eigenvalue spectrum is qualitatively different, as compared to Fig. 4(a) or (d), with respect to each of the properties noted above, *e.g.*, the ground state eigenvalue is not recognizably constant for $\xi < 1/5$, there is no apparent change in curvature at $\xi = 1/5$, and closer inspection reveals no level spacing compression of the type associated with the excited state quantum phase transition. This is already anticipated from the different eigenvalue range ($\sim 3N^2$) in the strong coupling limit, obtained in Sec. 5.5.

Similar distinctions between the large- N limit taken with $N \gg \Omega$ or $N \sim \Omega$ are obtained for the critical scaling properties, which we defer to a more comprehensive study. For now, we restrict attention to the basic energy spectra obtained with the present transitional Hamiltonian for the general two-level pairing model, and note that the spectrum for finite N depends strongly not just on the total degeneracy $n = n_1 + n_2$ of the two levels but on the equality or degree of inequality of the two level degeneracies n_1 and n_2 .

The transitional spectra for two different bosonic models with total degeneracy $n = 6$, and taken with $N = 10$ (*i.e.*, occupation substantially greater than the degeneracy), are compared in Fig. 5 (top): the *s-b* model ($n_1 = 1$ and $n_2 = 5$) [Fig. 5(a)] and the choice of two levels with equal degeneracies ($n_1 = 3$ and $n_2 = 3$) [Fig. 5(b)††]. The transitional spectra for the $(v_1 v_2) = (00)$ subspaces in Figs. 5(a) and (b) are similar

††Fig. 5(b) also corrects a labeling error in the legend of Fig. 6(c) of Ref. [14].

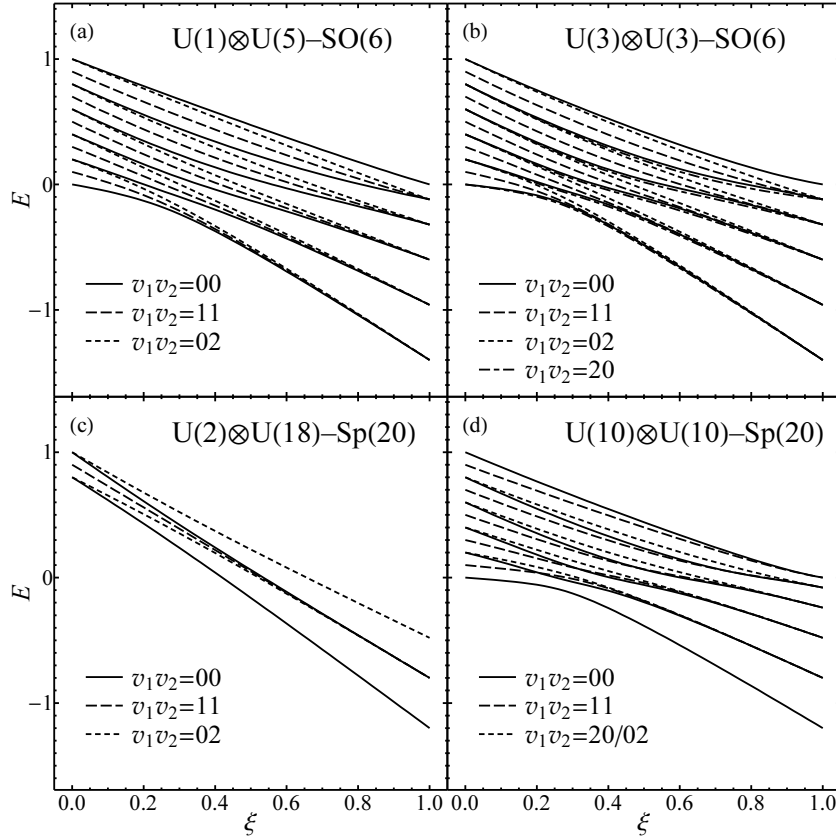


Figure 5. Eigenvalues for the bosonic two-level pairing model, with level degeneracies (a) $n_1 = 1$ and $n_2 = 5$ or (b) $n_1 = 3$ and $n_2 = 3$, and for the fermionic two-level pairing model, with level degeneracies (c) $n_1 = 2$ and $n_2 = 18$ or (d) $n_1 = 10$ and $n_2 = 10$, as functions of the control parameter ξ between the weak-coupling and strong-coupling limits. All calculations are for $N = 10$. Eigenvalues are shown only for the lowest-seniority subspaces (v_1v_2) , specifically, those with $v_1 + v_2 \leq 2$. The Hamiltonian H_+ of (69) is used for the bosonic calculations and H_- of (70) for the fermionic calculations.

to each other. Although only irreps of type $(0v_2)$ or $(1v_2)$ are obtained in the former case, more general irreps (v_1v_2) are possible in the latter case, naturally leading to a more complicated spectrum. In particular, it should be noted that the lowest state from each subspace of the form (v_10) approximately tracks the lowest (00) state in energy, and that these states are in fact lower in energy than the (00) state everywhere between the dynamical symmetry limits [see the lowest curve for $(v_1v_2) = (20)$ in Fig. 5(b)]. It is perhaps not surprising that, given a repulsive pairing interaction, the energy may be lowered by breaking pairs within the lower single-particle energy (*i.e.*, increasing v_1). In contrast, increasing v_2 also enforces nonzero occupation ($N_2 \geq v_2$) of the higher single-particle energy level and is therefore not as energetically preferred.

For the fermionic system, the difference between the transitional spectra for near-equal versus highly-imbalanced degeneracies for the two levels is marked. The transitional spectra for two different fermionic models with total degeneracy $n = 20$, again taken with $N = 10$ (which now represents half filling), are compared in

Fig. 5 (bottom): for the most extremely imbalanced possible choice of degeneracies ($n_1 = 2$ and $n_2 = 18$) [Fig. 5(c)] and with equal degeneracies ($n_1 = 10$ and $n_2 = 10$) [Fig. 5(d)]. The quantum phase transition which occurs for equal degeneracies is washed out in the limit of imbalanced degeneracies, as is evident in the simple, near-linear evolution of the ground state energy across Fig. 5(c). Such an effect may be expected on the basis of the Pauli principle. The lower level, of degeneracy $n_1 = 2$, easily saturates at full occupancy, so that the dynamics are effectively those of a one-level system of degeneracy $n_2 = 18$, which does not support critical phenomena as a function of pairing interaction strength.

7. Conclusion

Although the existence of duality relations between the number-conserving unitary and number-nonconserving quasispin algebras for the two-level system with pairing interactions is well known, and indeed these relations have proven useful in practical calculations for specific special cases of the two-level pairing model, here we have sought to establish a systematic treatment of the duality relations, both for bosonic and fermionic two-level pairing models and for arbitrary choice of level degeneracies. A principal goal has been to clarify the relationships between the disparate forms of the Hamiltonian encountered in the study and application of these models. The results are intended to provide a foundation for a more comprehensive investigation of quantum phase transitions in two-level pairing models — including the dependence of scaling properties on the bosonic or fermionic nature of the system and on the level degeneracies — beyond the special cases conventionally considered, namely, bosonic s - b models and fermionic models with equal degeneracy. The duality between orthogonal or symplectic algebras and the quasispin algebras is also relevant to the analysis of the classical dynamics of the system, through the associated coset spaces [41]. The dual algebras yield complementary descriptions involving classical coordinate spaces with different dimensionalities [42]. Finally, although the present derivations were given for the case of two-level models, they may readily be extended to the $SO(n_1 + n_2 + \dots)$ or $Sp(n_1 + n_2 + \dots)$ algebras associated with multi-level systems, with generators directly generalizing those of (14) and (15), for which the quantum phase transitions have been much less completely studied. Physical realizations of interest in this more general case include the nuclear shell model and descriptions of superconductivity in metallic grains.

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Appendix. Spherical tensor commutation relations

When working with angular momentum coupled products of spherical tensor operators, it is convenient to consider the *coupled commutator* [57, 72], itself a spherical tensor operator, with components given by

$$[A^{(a)}, B^{(b)}]_{\gamma}^{(c)} = \sum_{\alpha\beta} (a\alpha b\beta | c\gamma) [A_{\alpha}^{(a)}, B_{\beta}^{(b)}]. \quad (\text{A.1})$$

To clearly set out the identities used in establishing the commutators of the generators in Tables 4 and 5, the basic definitions and properties are summarized in this appendix. The use of coupled commutation results bypasses the tedious process of uncoupling the operators, *i.e.*, introducing multiple sums over products of Clebsch-Gordan coefficients, taking commutators of the spherical tensor components, and then recoupling.

If both bosonic and fermionic operators are to be considered, a consistent set of definitions is obtained if the quantity in brackets is taken to be the *graded* commutator, that is, either the commutator or the anticommutator according to the bosonic or fermionic nature of the operators. Specifically,

$$[A_{\alpha}^{(a)}, B_{\beta}^{(b)}] = A_{\alpha}^{(a)} B_{\beta}^{(b)} - \theta_{ab} B_{\beta}^{(b)} A_{\alpha}^{(a)}, \quad (\text{A.2})$$

where $\theta_{ab} = +$ if either A or B is a bosonic operator, and $\theta_{ab} = -$ if both A and B are fermionic operators. (For the sake of these definitions, it is assumed that a bosonic operator has integer angular momentum and a fermionic operator has half-integer angular momentum.) The coupled commutator can be written directly in terms of coupled products as

$$[A^{(a)}, B^{(b)}]^{(c)} = (A^{(a)} \times B^{(b)})^{(c)} - \theta_{ab} (-)^{c-a-b} (B^{(b)} \times A^{(a)})^{(c)} \quad (\text{A.3})$$

and obeys the symmetry or antisymmetry relation

$$[B^{(b)}, A^{(a)}]^{(c)} = -\theta_{ab} (-)^{c-a-b} [A^{(a)}, B^{(b)}]^{(c)}. \quad (\text{A.4})$$

The uncoupled commutators of the spherical tensor components may be recovered from the coupled commutators, if needed, by inverting (A.1) to give

$$[A_{\alpha}^{(a)}, B_{\beta}^{(b)}] = \sum_{c\gamma} (a\alpha b\beta | c\gamma) [A^{(a)}, B^{(b)}]_{\gamma}^{(c)}. \quad (\text{A.5})$$

The product rule for coupled commutators is [57, (6)]

$$\begin{aligned} [(A \times B)^{(e)}, C]^{(d)} &= \sum_f \hat{e} \hat{f} [(-)^{a+b+c+d} \left\{ \begin{matrix} a & b & e \\ c & d & f \end{matrix} \right\} (A \times [B, C]^{(f)})^{(d)} \\ &\quad + \theta_{bc} \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} ([A, C]^{(f)} \times B)^{(d)}]. \end{aligned} \quad (\text{A.6})$$

A second application of this identity yields the double product rule needed for evaluating commutators of one-body or pair operators,

$$\begin{aligned} [(A \times B)^{(e)}, (C \times D)^{(f)}]^{(g)} \\ = \sum_{hk} \hat{e} \hat{f} \hat{h} \hat{k} [(-)^{e+c+d+g} (-)^{a+b+c+h} \left\{ \begin{matrix} c & d & f \\ g & e & h \end{matrix} \right\} \left\{ \begin{matrix} a & b & e \\ c & h & k \end{matrix} \right\} [(A \times [B, C]^{(k)})^{(h)} \times D]^{(g)} \end{aligned}$$

$$\begin{aligned}
& +\theta_{bc}(-)^{e+c+d+g}(-)^{b+c+e+k}\left\{\begin{matrix} c & d & f \\ g & e & h \end{matrix}\right\}\left\{\begin{matrix} a & b & e \\ h & c & k \end{matrix}\right\}[(A, C]^{(k)} \times B)^{(h)} \times D]^{(g)} \\
& +\theta_{ec}(-)^{e+c+f+h}(-)^{a+b+d+h}\left\{\begin{matrix} c & d & f \\ e & g & h \end{matrix}\right\}\left\{\begin{matrix} a & b & e \\ d & h & k \end{matrix}\right\}[C \times (A \times [B, D]^{(k)})^{(h)}]^{(g)} \\
& +\theta_{ec}\theta_{bd}(-)^{e+c+f+h}(-)^{b+d+e+k}\left\{\begin{matrix} c & d & f \\ e & g & h \end{matrix}\right\}\left\{\begin{matrix} a & b & e \\ h & d & k \end{matrix}\right\}[C \times ([A, D]^{(k)} \times B)^{(h)}]^{(g)}. \tag{A.7}
\end{aligned}$$

If operators A^\dagger and B^\dagger are creation operators, obeying canonical commutation or anticommutation relations, the canonical commutators are represented in coupled form by [57, (10)]

$$[\tilde{A}, B^\dagger]^{(c)} = \hat{a}\delta_{AB}\delta_{c0} \tag{A.8}$$

and $[\tilde{A}, \tilde{B}]^{(c)} = [A^\dagger, B^\dagger]^{(c)} = 0$. The coupled commutator of two one-body operators is therefore

$$\begin{aligned}
& [(A^\dagger \times \tilde{B})^{(e)}, (C^\dagger \times \tilde{D})^{(f)}]^{(g)} = (-)^{2b}(-)^{a+d+g}\hat{e}\hat{f}\left\{\begin{matrix} e & f & g \\ d & a & b \end{matrix}\right\}(A^\dagger \times \tilde{D})^{(g)}\delta_{BC} \\
& -\theta_{ab}\theta_{bc}\theta_{ca}(-)^{b+c+e+f}\hat{e}\hat{f}\left\{\begin{matrix} e & f & g \\ c & b & a \end{matrix}\right\}(C^\dagger \times \tilde{B})^{(g)}\delta_{AD}, \tag{A.9}
\end{aligned}$$

as needed, *e.g.*, for the commutators of the generators of $U(n_1 + n_2)$.

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